1.16 Factors, Multiples, Prime Numbers and Divisibility

- **Factor** – an integer that goes into another integer exactly without any remainder.
  Need to be able to find them *all* for a particular integer – it's usually best to start at 1 and find them in pairs. Write them at opposite ends of the page so that you have them in order when you've finished.
  e.g., for 20 begin **1, 2, , 10, 20**
  You need only try every number up to the square root of the number n whose factors you're finding. Any number bigger than $\sqrt{n}$ would already have been found because its partner would be smaller than $\sqrt{n}$.
- **Multiple** – a number in the times table.
- **Prime Number** – a number with exactly *two factors* (1 and itself).
  By this definition 1 isn't prime because it hasn't got enough factors.
- Factors and multiples are opposites in the sense that if a is a factor of b then b is a multiple of a.
- Another difference is that all numbers have a finite number of factors but an infinite number of multiples.
- It can be useful to find the HCF and LCM by listing the factors/multiples before moving on to using prime factorisation to do it. It's helpful to find both the HCF and the LCM for the same pairs of numbers so that pupils are less likely to muddle up the two processes.
  HCF must be less than or equal to the smaller/smallest of the numbers; LCM must be more than or equal to the larger/largest of the numbers.

1.16.1	Find a square number that has fewer than 3 factors. Find an even number that is prime.	*Answers: 1 (only one factor –itself)* *2 only*
1.16.2	Shade in on a 100 square (see sheet) all the even numbers and the multiples of 3, 5 and 7 (not including 2, 3, 5 and 7 themselves). The prime numbers are left. There are infinitely many prime numbers. Euclid (about 330-270 BC) proved so by contradiction: Imagine there are only a certain number. Imagine multiplying them all together and adding 1. This new number either is prime (contradiction) or else it has a prime factor. But none of the prime numbers you've already got can be factors (all go in with remainder 1), so contradiction again. Therefore, proved.	*See 100-square sheet.* *There are 25 of them.* *It's useful to have a list of prime numbers less than 100 by this means or some other (see sheet suitable for sticking into exercise books).* *You only need to go up to multiples of 7 because 11 is the next prime number and* $11^2 > 100$. *This process is sometimes called the "Sieve of Eratosthenes" (about 280 BC – 190 AD).* *The logic of proof by contradiction can be appealing.*
1.16.3	**NEED** scrap paper (A4). Give each pupil (or pair) a sheet of scrap paper. Fold into 20 pieces (5 × 4, not as hard as it sounds!), tear up, write on the numbers 1-20. Then, "Show me the multiples of 4", and pupils push those numbers into one spot. "Show me the prime numbers greater than 7", etc.	*This doesn't need to be done too carefully.* *It's easy for the teacher to see how everyone is doing.* Can use this method for other number work; e.g., "Show me four numbers that add up to 15", or "four numbers with a mean of 8."
1.16.4	Draw a Venn diagram with subsets like "multiples of 4", "even numbers", "factors of 18", "prime numbers", etc.	*Could extend to probability; e.g., "If you choose an integer at random from the set 1-20, what is the probability of choosing an even prime number?"* *Answer* $\frac{1}{20}$.
1.16.5	Find a number with only 1 factor. Find some numbers with exactly 2 factors. Find some numbers with exactly 3 factors. What's special about them?	*Answers: only number 1* *2, 3, 5, 7, etc. (prime numbers)* *4, 9, 25, etc.* *They're squares of prime numbers.* *(If* p *is the prime number then the factors of* p^2 *are* 1, p *and* p^2.*)*

Square numbers have an odd number of factors because they have a "repeated" factor.

What kind of numbers have n factors?
How can you decide how many factors a number will have without working them all out?

Suppose that p, q, r, ... are prime numbers, and a, b, c, ... are integers ≥ 0. Any power of a prime number can be written as p^a, and will have $a+1$ factors ($1, p, p^2, p^3, \ldots p^a$). Hence p has 2 factors and p^2 has 3 factors, as above. Since every number can be factorised into primes, every number x can be written as $p^a q^b r^c \ldots$ and will have $(a+1)(b+1)(c+1)\ldots$ factors, since any of the $a+1$ powers of p can be multiplied by any of the powers of the others.
So the possibilities are as below.

You can set a challenge such as finding a number with exactly 13 factors.
One answer would be p^{12} (where p is a prime number), so choosing $p=3$ gives the number 531 441.
The factors are 1, 3, 9, 27, 81, 243, 729, 2 187, 6 561, 19 683, 59 049, 177 147 and 531 441, and there are thirteen of them.

no. of factors	***prime factorisations of numbers that have that many factors***
1	1
2	p
3	p^2
4	p^3 *or* pq
5	p^4
6	p^5 *or* p^2q
7	p^6
8	p^7 *or* p^3q *or* pqr
9	p^8 *or* p^2q^2
10	p^9 *or* p^4q

So it's to do with the number of factors that n has (and n is how many factors the original number has)!

1.16.6 For prime factorisation, it's possible to draw tree diagrams going down the page.
Stop and put a ring around it when you reach a prime number.
Try each time to split the number into two factors that are as nearly equal as possible, because that leads to fewer steps.

e.g., for 24

```
            24
           /   \
         6       4
        / \     / \
       2   3   2   2
```

so 24 = $2^3 \times 3$

Could discuss why we don't count 1 as a prime number.

One reason is that prime factorisation would go on for ever! It's useful to have one unique way (apart from the order you write it in) of prime factorising every integer.

1.16.7 Tests for divisibility. Build up a big table of all the tests on the board (see sheet).
Then make 3 columns headed by 3 "random" numbers (e.g., 2016, 3465 and 2400) and put the numbers 1-12 down the side. Place either a tick or a cross in each column to say whether the column number is divisible by the row number.

It's productive to look for patterns in the answers (e.g., if a number is divisible by 10 then it's necessarily divisible by 2 and by 5, etc.).

Note for example that divisible by 4 $\Rightarrow$ divisible by 2, but not the other way round.

1.16.8 "I come into the classroom and ask the class to get into pairs – but 1 person is left over.
We have to have groups of equal size, so I say never mind, instead get into groups of 3. Again 1 person is left over.
We try groups of 4. Again, 1 person is left over.
Finally groups of 5 works. How many people do you think there are in the class?"

Answer: 25 pupils; 85, 145, 205, etc. also work but hopefully would not be real class sizes!

One strategy is to say that $n-1$ must be a multiple of 2, 3 and 4; i.e., a multiple of 12, and go through the multiples of 12 until you find one where the number that is one higher is a multiple of 5.

1.16.9 "I'm thinking of a number …" or "I'm thinking of

Lots of possibilities.

two numbers …"

1.16.10 Which number less than 100 has the most factors?

Answer: 96 (= $2^5 \times 3$), so 12 (= 6 × 2) factors: 1, 2, 3, 4, 6, 8, 12, 16, 24, 32, 48, 96.

Which has the fewest?

Answer: 1 (just has itself)

1.16.11 Perfect Numbers.
Classify each integer up to 20 as *perfect* (if it is the sum of all its factors apart from itself), *abundant* (if it is less than the sum of all its factors apart from itself) or *deficient* (if it is more). e.g., 6 = 1 + 2 + 3 so it's a perfect number, whereas the factors of 8 (apart from 8) add up to only 1 + 2 + 4 = 7, so it's deficient (but only just).

no.		***no.***		***no.***		***no.***	
1	*d*	***6***	*p*	***11***	*d*	***16***	*d*
2	*d*	***7***	*d*	***12***	*a*	***17***	*d*
3	*d*	***8***	*d*	***13***	*d*	***18***	*a*
4	*d*	***9***	*d*	***14***	*d*	***19***	*d*
5	*d*	***10***	*d*	***15***	*d*	***20***	*a*

Most numbers are deficient, because they don't have enough factors. Primes and powers of primes are all deficient, although powers of 2 are almost perfect (just too small by 1).
Perfect numbers are 6, 28, 496, 8218, etc.
Abundant numbers are 12, 18, 20, 24, 30, 36, etc.
Any multiple of a perfect or abundant number is abundant, and any factor of a perfect or deficient number is deficient.
No-one knows if there are any odd perfect numbers, but they've checked up to 10^{300} and haven't found any!

Why do you think we don't include the number itself when we add up the factors?
Because then you'd always get too much and it wouldn't be very interesting!

1.16.12 Which two numbers, neither containing any zeroes, multiply to make 100?

Answers: 4 and 25

1000?

8 and 125

1 000 000?

64 and 15625

In general, $10^x = 2^x \times 5^x$.

1.16.13 There are three brothers. The first one comes home 1 day in every 6 days, the second one once every 5 days and the third one once every 4 days. How often will they all be together?

Answer: The LCM of 6, 5 and 4 is 60.
So they'll meet every 60 days.

This assumes that the 1st and 3rd brothers don't perpetually miss each other (i.e., they start off all together). The numbers have to be pairwise co-prime to guarantee that they will all meet.

1.16.14 What is the smallest integer that all of the integers 1, 2, 3, 4, 5, 6, 7, 8 and 9 will go into?

Answer: The LCM of all the numbers 1 to 9 is $2^3 \times 3^2 \times 5 \times 7 = 2520$.

1.16.15 What is special about this number?
3816547290

Answer: Working from the left, the first digit is divisible by 1, the first two digits together (38) are divisible by 2, the first three digits are divisible by 3, etc. (And it uses the digits from 0 to 9 once each.)

Can you invent another number like it?

1.16.16 What collection of positive integers that add up to 100 (repeats allowed) makes the largest possible product when they are all multiplied together?

Answer: use two 2's and thirty-two 3's, and you get a total of 100 and a product of $2^2 \times 3^{32}$, which is about 7×10^{15}.
The logic behind this is that two 3's have a bigger product (9) than three 2's (8), so you would rather use 3's than 2's, but since 6's don't go into 100 you have to use two 2's to make the sum of 100.
Obviously you wouldn't use any 1's, since they wouldn't increase the product. There's no advantage using 4's, because 2 + 2 = 4 and 2 × 2 = 4. For any n bigger than 4, $2(n-2) > 4$, so it's better to split up these numbers into two's.

What about other totals? e.g., 16?

Answer: $3^4 \times 2^2 = 324$ *by the same logic.*

A related problem is to find the maximum product of *two numbers* (not necessarily integers) that sum to 100.
e.g., 10 + 90 = 100 and 10 × 90 = 900.
Can you split up 100 into two numbers that have a bigger product than this?

Answer: If the smaller of the two numbers is x *, then the product* p *will be* $p = x(100 - x)$ *and this is a quadratic function with a maximum value of 2500 when* $x = 50$ *.*
In general, if the total has to be t *then the maximum product is* $\frac{t}{2} \times \frac{t}{2} = \frac{t^2}{4}$ *.*

What about multiplying together three or more numbers that sum to 100?

If n *numbers have a total of* t *, then their product will be a maximum when each of them is* $\frac{t}{n}$ *, in which case their product will be* $\left(\frac{t}{n}\right)^n$ *.*

1.16.17 Prove that all prime numbers > 3 are either one more or one less than a multiple of six.

This is a kind of proof by exhaustion. *We say that all integers can be written in one of 6 ways and we list them. Then we exclude types of number that we know could never be prime. All prime numbers must fit one of the two options left.*

Answer: Every number can be written as $6n$ *or* $6n+1$ *or* $6n+2$ *or* $6n+3$ *or* $6n+4$ *or* $6n+5$ *.*
($6n+6$ *would be a multiple of 6, so could be written as just* $6n$ *, etc.).*
Two's go into $6n$ *,* $6n+2$ *and* $6n+4$ *(since both terms are multiples of 2), and threes go into* $6n$ *and* $6n+3$ *. So the only numbers that* don't *have factors of 2 or 3 are* $6n+1$ *and* $6n+5$ *. Not all of these will be prime, of course (e.g., 25 is* $6n+1$ *), but all prime numbers (apart from 2 and 3) must take one of these forms.*

1.16.18 Make a chart to show factors of numbers up to, say, 20. Shade in the factors.
Look for patterns.

How will the pattern continue?

factor

no.	***1***	***2***	***3***	***4***	***5***	***6***	***7***	***8***	***9***	***10***
1	■									
2	■	■								
3	■		■							
4	■	■		■						
5	■				■					
6	■	■	■			■				
7	■						■			
8	■	■		■				■		
9	■		■						■	
10	■	■			■					■

1.16.19 Think of an integer between 1 and 9.
Add 1. Multiply by 9. Add up the digits. Divide by 3. Add 1. What do you get?

Always 4. Why?

The digit sum of $9n$ *where* n *is an integer* ≤ 10 *is always 9. So when you divide by 3 and add 1 you will always get 4.*

1.16.20 Prisoners. A hundred prisoners are locked in their cells (numbered 1 to 100). The cell doors have special handles on the outside. When a guard turns the handle a locked door is unlocked and an unlocked door is locked.
All the doors start off locked. Then prison guard number 1 comes along and turns the handle of every door once (so all the doors are unlocked). Guard number 2 comes along and turns the handle of every *second* door (starting with cell 2). Guard 3 turns every third handle, and so on.
After the 100th guard has been past, which prisoners can now get out of their cells?

Answer: 1, 4, 9, 16, 25, 36, 49, 64, 81, 100
The answer is the square numbers since they have an odd number of factors, so their handles will have been turned an odd number of times.

Could try with a smaller number of cubes or pennies (heads up for "unlocked", tails for "locked").

1.16.21 Stamps. I have an unlimited number of 2p and 3p

Answers: With co-prime (HCF=1) values you can stamps. What possible (integer) values can I make?

make all possible values beyond a certain number; e.g., with 2p and 3p, only 1 is impossible. Clearly 2p stamps enable all even amounts, and one 3p stamp plus any number of 2p stamps enable all odd amounts 3 or more. Hence everything except 1p.

What about with 3p and 4p stamps?

Harder now. Consider 3p stamps (since 3<4). If at any stage I can make 3 consecutive numbers, then from then on I can have any amount, by adding 3's to each. You can do 6 (= 3 + 3), 7 (= 3 + 4) and 8 (= 4 + 4), so the only impossibles are 1, 2 and 5.

What about 3p and 5p stamps?

Again, the first 3 consecutives you can make are 8 (3+5), 9 (3+3+3) and 10 (5+5), so the only impossibles are 1, 2, 4 and 7.

Variations on this include coins (obviously); fitting kitchen cabinets into different length kitchens; and scores with an unlimited number of darts on dartboards that have just a few different sections.

In general where x and y are co-prime, the highest impossible amount is $xy-(x+y)$.
(This is hard to prove, but see sheet.)

Functions like $A=3x+5y$ where x and y are integers are called Diophantine equations *(Diophantus, about 200-280 AD). A can be any integer if the co-efficients (3 and 5) are co-prime (HCF=1), but x and y may need to be negative.*

This is a very rich investigation which can run and run. It can turn into a Fibonacci-type problem by asking how many ways there are of making, say, 50p from 5p and 7p stamps. You can let the order matter by saying that you are buying the stamps at a post office and the assistant is annoyingly pushing the stamps under the safety screen one at a time.
So 5p + 7p and 7p + 5p count as different ways of making 12p.
A spreadsheet might help.

Answer: 57 ways.
The key thing to notice is that if n_c is the number of ways of making a total of c pence out of x pence and y pence stamps, then assuming that $c>x$ and $c>y$, $n_c = n_{c-x} + n_{c-y}$. (because you add one x pence or one y pence stamp). For example, if $x=5$ and $y=7$ then $n_{17}=3$ and $n_{19}=3$ so $n_{24}=6$.

You could make a tree diagram starting ...

```
          50
        /    \
      43      45
     /  \    /  \
   36   38  38   40
```

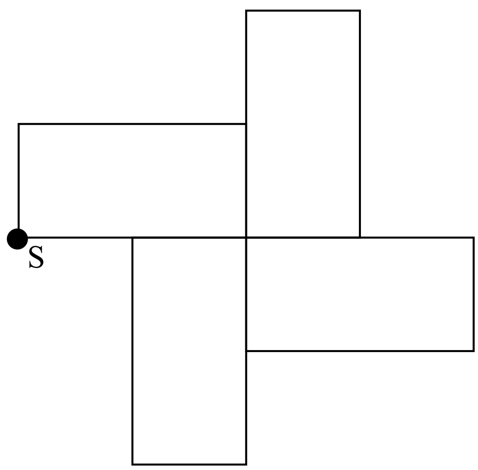
... to count up the number of ways.

(See the table of answers on the sheet – highest impossible amounts are shaded in.)

1.16.22 Spirolaterals.
Make a Vedic Square by writing the numbers 1 to 9 around the edge of an 8 × 8 square, multiplying and filling in the boxes.
Then re-draw it finding the *digit sums* of the answers.
This gives

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
2	4	6	8	1	3	5	7	9
3	6	9	3	6	9	3	6	9
4	8	3	7	2	6	1	5	9
5	1	6	2	7	3	8	4	9
6	3	9	6	3	9	6	3	9
7	5	3	1	8	6	4	2	9
8	7	6	5	4	3	2	1	9

Islamic art is based on Vedic squares.
(Digit sum = *sum of the digits of the numbers.)*



9	9	9	9	9	9	9	9	9

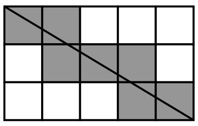
Then begin somewhere in the middle of a sheet of A4 0.5 cm × 0.5 cm squared paper.
Mark a dot. Choose a column in the table, say the 3's. Draw a line 3 squares long straight up the paper, turn right, draw a line 6 squares long, turn right, then 9, then 3, and so on down the column, always turning *right* (see drawing on the right – S is the starting point).
When you get to the bottom of a column of numbers, just start again at the top.
Keep going until you get back to the place where you started.

The easiest columns to do are 3, 6, 9 (but 9 is boring – a square, obviously).

All of them will fit on A4 0.5 cm × 0.5 cm squared paper, but you need to start in a sensible place. Teacher may need to assist.

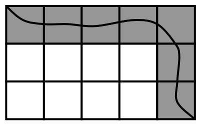
The tricky thing is seeing that turning right when coming down *the page towards you means actually going left. Some people find it helpful to turn the paper as you go (like turning a map), so you're always drawing away from you, and some people just find that confusing. It's also useful to mention the name "*spiro*lateral" so we expect it to "spiral in", not meander further and further towards the edge of the paper!*

1.16.23 Diagonals in Rectangles.
If an $x \times y$ rectangle is drawn on squared paper, how many squares does the diagonal line pass through?
x and y are integers.
e.g., for a 5×3 rectangle, the line goes through 7 squares.



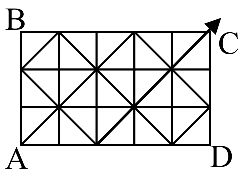
We count it as "going through" a square even if it only *just* clips the square. Only if it goes *exactly* through a crossing-point do we not count the square.

Answer:
If x and y are co-prime (HCF=1) then the line must go through $x+y-1$ squares. This is the smallest number of squares between opposite corners. (Imagine a curly line – see below).



If $HCF(x,y)>1$ *then every* $\frac{1}{HCF(x,y)}$ *of the way along the diagonal line will be a crossing point, since* $\frac{x}{HCF(x,y)}$ *and* $\frac{y}{HCF(x,y)}$ *will be integers.*
Each of these $HCF(x,y)-1$ *crossing points means one fewer square for the diagonal to go through. So that means the total number of squares will be*
$x+y-1-(HCF(x,y)-1)$
$=x+y-HCF(x,y)$

1.16.24 Snooker investigation.
A snooker ball is projected from the near left corner (A, below) of a rectangular snooker table at 45° to the sides. If there are pockets at all four corners, and the table has dimensions $x \times y$ (x is the horizontal width), which pocket will the ball end up in?
x and y are integers.



Assume that every time the ball hits a side it rebounds at 45°, and that the ball never runs out of kinetic energy.

Above for a 5×3 table, the answer is pocket C.

Try some examples on squared paper.

Answer:
Every diagonal step forward moves the ball 1 square horizontally and 1 square vertically. Since x *is the horizontal distance, after* $x, 3x, 5x, ...$ *"steps" the ball will be at the right wall. After* $2x, 4x, 6x, ...$ *steps the ball will be at the left wall.*
Similarly, after $y, 3y, 5y, ...$ *steps the ball will be at the top side. After* $2y, 4y, 6y, ...$ *steps the ball will be at the bottom side.*
Therefore, the first time that a multiple of x *is equal to a multiple of* y *(i.e., after* $LCM(x,y)$ *steps), the ball will be in one of the corners.*

It will never be corner A, because the ball only reaches there if it travels an even number of x's *and an even number of* y's. *That will never be the LCM of* x *and* y *because half that many* x's *would match half that many* y's.

If $\frac{LCM(x,y)}{x}$ *is odd, then the pocket will be either*

C or D. Otherwise, it will be A or B.

If $\frac{LCM(x, y)}{y}$ *is odd, then the pocket will be either B or C. Otherwise, it will be A or D.*

Taken together, this means you can always predict which pocket the ball will end up in.

Hint:
How many diagonal steps will the ball move before it lands in a pocket?

e.g., for a 10×4 *table,* $LCM(10,4) = 20$

$\frac{20}{10}$ *is even so AB side;* $\frac{20}{4}$ *is odd so BC side. Hence pocket B.*

1.16.25 What is the significance of the digit sum of an integer when the integer isn't a multiple of 9? Try working some out.

Answer: It's the remainder when you divide the number by 9. E.g., $382 \div 9 = 42$ *, remainder 4. And the digit sum of 382 is 4 (actually 13, but the digit sum of 13 is 4).*
(The digit sum of a multiple of 9 is itself a multiple of 9.)

Find out what "casting out 9's" refers to. (Suitable for a homework: ask grandparents.)

You can prove that the digit sum of a 3-digit number, say, is the remainder when dividing by 9 by writing "abc", as $n = 100a + 10b + c$. (This process would work just as well however many digits the number had.)

$$n = 100a + 10b + c$$
$$= 99a + 9b + a + b + c$$
$$= 9(11a + b) + (a + b + c)$$

so $a + b + c$ is the remainder when n is divided by 9. (This assumes that $a + b + c < 9$. If it's *equal* to 9, then n is a multiple of 9; if it's *more than* 9, then we can just start again and find the digit sum of *this* number, because the remainder of this number when divided by 9 will be the same as the remainder of n when divided by 9.)

Hence the method of casting out 9's" in which every integer in the calculation is replaced by its digit sum. When the calculation is done with these numbers, the answer should be the digit sum of the answer to the original question.
This provides a way of checking.
e.g., $946 + 326 = 1272$ *, replacing 946 and 326 by their digit sums gives* $1 + 2 = 3$ *, and 3 is the digit sum of 1272. This doesn't guarantee that the sum is correct, but if this test* doesn't *work then the sum is* definitely *wrong.*

This is a bit like the modern "check-sums" method used on bar-codes to make sure the machine has read the numbers accurately. Here a single mistake can always be identified.

1.16.26 Why not define "highest common multiple" and "lowest common factor", as well?

Answer: If you think about it, LCF would always be 1 and HCM would always be infinite!

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Prime numbers less than 100

2	3	5	7	11
13	17	19	23	29
31	37	41	43	47
53	59	61	67	71
73	79	83	89	97

Prime numbers less than 100

2	3	5	7	11
13	17	19	23	29
31	37	41	43	47
53	59	61	67	71
73	79	83	89	97

Prime numbers less than 100

2	3	5	7	11
13	17	19	23	29
31	37	41	43	47
53	59	61	67	71
73	79	83	89	97

Prime numbers less than 100

2	3	5	7	11
13	17	19	23	29
31	37	41	43	47
53	59	61	67	71
73	79	83	89	97

Prime numbers less than 100

2	3	5	7	11
13	17	19	23	29
31	37	41	43	47
53	59	61	67	71
73	79	83	89	97

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31	37	41	43	47
53	59	61	67	71
73	79	83	89	97

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2	3	5	7	11
13	17	19	23	29
31	37	41	43	47
53	59	61	67	71
73	79	83	89	97

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2	3	5	7	11
13	17	19	23	29
31	37	41	43	47
53	59	61	67	71
73	79	83	89	97

Prime numbers less than 100

2	3	5	7	11
13	17	19	23	29
31	37	41	43	47
53	59	61	67	71
73	79	83	89	97

Prime numbers less than 100

2	3	5	7	11
13	17	19	23	29
31	37	41	43	47
53	59	61	67	71
73	79	83	89	97

Prime Numbers

2	3	5	7	11	13	17	19	23	29
31	37	41	43	47	53	59	61	67	71
73	79	83	89	97	101	103	107	109	113
127	131	137	139	149	151	157	163	167	173
179	181	191	193	197	199	211	223	227	229
233	239	241	251	257	263	269	271	277	281
283	293	307	311	313	317	331	337	347	349
353	359	367	373	379	383	389	397	401	409
419	421	431	433	439	443	449	457	461	463
467	479	487	491	499	503	509	521	523	541
547	557	563	569	571	577	587	593	599	601
607	613	617	619	631	641	643	647	653	659
661	673	677	683	691	701	709	719	727	733
739	743	751	757	761	769	773	787	797	809
811	821	823	827	829	839	853	857	859	863
877	881	883	887	907	911	919	929	937	941
947	953	967	971	977	983	991	997	1009	1013
1019	1021	1031	1033	1039	1049	1051	1061	1063	1069
1087	1091	1093	1097	1103	1109	1117	1123	1129	1151
1153	1163	1171	1181	1187	1193	1201	1213	1217	1223
1229	1231	1237	1249	1259	1277	1279	1283	1289	1291
1297	1301	1303	1307	1319	1321	1327	1361	1367	1373
1381	1399	1409	1423	1427	1429	1433	1439	1447	1451
1453	1459	1471	1481	1483	1487	1489	1493	1499	1511
1523	1531	1543	1549	1553	1559	1567	1571	1579	1583
1597	1601	1607	1609	1613	1619	1621	1627	1637	1657
1663	1667	1669	1693	1697	1699	1709	1721	1723	1733
1741	1747	1753	1759	1777	1783	1787	1789	1801	1811
1823	1831	1847	1861	1867	1871	1873	1877	1879	1889
1901	1907	1913	1931	1933	1949	1951	1973	1979	1987
1993	1997	1999	2003	2011	2017	2027	2029	2039	2053
2063	2069	2081	2083	2087	2089	2099	2111	2113	2129
2131	2137	2141	2143	2153	2161	2179	2203	2207	2213
2221	2237	2239	2243	2251	2267	2269	2273	2281	2287
2293	2297	2309	2311	2333	2339	2341	2347	2351	2357
2371	2377	2381	2383	2389	2393	2399	2411	2417	2423
2437	2441	2447	2459	2467	2473	2477	2503	2521	2531
2539	2543	2549	2551	2557	2579	2591	2593	2609	2617
2621	2633	2647	2657	2659	2663	2671	2677	2683	2687
2689	2693	2699	2707	2711	2713	2719	2729	2731	2741
2749	2753	2767	2777	2789	2791	2797	2801	2803	2819
2833	2837	2843	2851	2857	2861	2879	2887	2897	2903
2909	2917	2927	2939	2953	2957	2963	2969	2971	2999
3001	3011	3019	3023	3037	3041	3049	3061	3067	3079
3083	3089	3109	3119	3121	3137	3163	3167	3169	3181
3187	3191	3203	3209	3217	3221	3229	3251	3253	3257
3259	3271	3299	3301	3307	3313	3319	3323	3329	3331
3343	3347	3359	3361	3371	3373	3389	3391	3407	3413
3433	3449	3457	3461	3463	3467	3469	3491	3499	3511
3517	3527	3529	3533	3539	3541	3547	3557	3559	3571
3581	3583	3593	3607	3613	3617	3623	3631	3637	3643
3659	3671	3673	3677	3691	3697	3701	3709	3719	3727
3733	3739	3761	3767	3769	3779	3793	3797	3803	3821
3823	3833	3847	3851	3853	3863	3877	3881	3889	3907
3911	3917	3919	3923	3929	3931	3943	3947	3967	3989
4001	4003	4007	4013	4019	4021	4027	4049	4051	4057

Tests for Divisibility

integer	test	divides into the number if…
1	no test	any integer
2	look at units digit	0, 2, 4, 6 or 8
3	digit sum →→	3, 6, 9
4	look at last 2 digits	divisible by 4
5	look at units digit	0 or 5
6	test for 2 and test for 3	passes both tests
7	double the units digit and subtract from the rest of the number →→	divisible by 7
8	divide it by 2	divisible by 4
9	digit sum →→	9
10	look at units digit	0
11	alternating digit sum (…+–+–+)→→	0

Notes

- →→ means do the same thing to the answer and keep going until you have only 1 digit left
- "digit sum →→" on 732 gives 732 → 12 → 3 (so passes the test for 3)
- "alternating digit sum (…+–+–+)→→" on 698786 gives + 6 – 8 + 7 – 8 + 9 – 6 (working from *right to left* and beginning with +) = 0 (so passes the test for 11)
- "double the units digit and subtract from the rest of the number →→" on 39396 gives 3939 – 12 = 3927 → 392 – 14 = 378 → 37 – 16 = 21, divisible by 7 (so passes the test for 7)

- You can use a calculator to generate multiples to use for practice. (E.g., type in a 4-digit "random" integer, multiply by 7 and you have a "random" multiple of 7 for trying out the test for divisibility by 7.)
- A test for divisibility by 12 could be to pass the tests for divisibility by 3 and by 4. It wouldn't be any good to use the tests for divisibility by 2 and by 6 because passing the test for 6 means that the number must be even so the test for 2 adds nothing. 3 and 4 are co-prime (HCF = 1), but 6 and 2 aren't.

These tests are worth memorising and practising.

Stamps Investigation (Proof)

If the stamps are x pence and y pence (x and y co-prime), then the highest impossible value (with unlimited quantities of each) is $xy - x - y$ pence.

This is easy to show for a particular pair of stamp values.

For example, for $x = 3$ and $y = 5$ we write down all the positive integers using three columns (see right – imagine the columns going down for ever).

1	2	3
4	5	6
7	8	9
10	11	12
13	14	15
16	17	18
…	…	…

We're going to shade in all the numbers that are *possible*.
Clearly the right column will all be possible by using just the 3p stamps, because the numbers are all multiples of 3.

In the second row, 5 will be possible (one 5p stamp) so we shade that in. Also everything under 5 in the second column will be possible by using one 5p stamp and different numbers of 3p stamps.

Similarly, 10 (two 5p stamps) and everything below it will be possible.
So 7 is the highest impossible amount (and this is $3 \times 5 - 3 - 5$).

Now we try the same thing generally, with x pence and y pence stamps (x and y co-prime).

The diagram is shown on the right.
We will assume that $x < y$.
As before, the right column will all be possible (multiples of x pence), so we shade it in.

1	2	3	…	$x-1$	x
$x+1$	$x+2$	$x+3$	…	$2x-1$	$2x$
$2x+1$	$2x+2$	$2x+3$	…	$3x-1$	$3x$
…	…	…	…	…	…

The column containing y will all be possible from y downwards (y, $y+x$, $y+2x$, …) and this cannot be the same column as the x column (x, $2x$, $3x$, …) because y is not a multiple of x (they're co-prime).

For the same reason, $2y$ cannot be in either of the two columns dealt with so far (unless there are only two columns because $x = 2$). (It can't be in the "x column" because $2y$ cannot be a multiple of x, and the extra y places moved on from y can't equal a multiple of x, which would be necessary if it were in the same column as y).
So we keep locating the next multiple of y, and we always find it in a previously unvisited column, and we shade it in and we shade in the rest of that column below it.

There are x columns altogether, so when we reach $(x-1)y$ and shade that in (and all the numbers beneath it) the highest impossible amount will be the number directly above $(x-1)y$ (since all the other numbers in that row will already be shaded in).
This number will be $(x-1)y - x = xy - x - y$, and so this will be the highest impossible value.

Number of ways n_c of making up a cost c pence out of stamps

c	5p, 7p	3p, 7p	5p, 8p	5p, 9p	7p, 10p
1	0	0	0	0	0
2	0	0	0	0	0
3	0	1	0	0	0
4	0	0	0	0	0
5	1	0	1	1	0
6	0	1	0	0	0
7	1	1	0	0	1
8	0	0	1	0	0
9	0	1	0	1	0
10	1	2	1	1	1
11	0	0	0	0	0
12	2	1	0	0	0
13	0	3	2	0	0
14	1	1	0	2	1
15	1	1	1	1	0
16	0	4	1	0	0
17	3	3	0	0	2
18	0	1	3	1	0
19	3	5	0	3	0
20	1	6	1	1	1
21	1	2	3	0	1
22	4	6	0	0	0
23	0	10	4	3	0
24	6	5	1	4	3
25	1	7	1	1	0
26	4	15	6	0	0
27	5	11	0	1	3
28	1	9	5	6	1
29	10	21	4	5	0
30	1	21	1	1	1
31	10	14	10	0	4
32	6	28	1	4	0
33	5	36	6	10	0
34	15	25	10	6	6
35	2	37	1	1	1
36	20	57	15	1	0
37	7	46	5	10	4
38	15	51	7	15	5
39	21	85	20	7	0
40	7	82	2	1	1
41	35	76	21	5	10
42	9	122	15	20	1
43	35	139	8	21	0
44	28	122	35	8	10
45	22	173	7	2	6
46	56	224	28	15	0
47	16	204	35	35	5
48	70	249	10	28	15
49	37	346	56	9	1
50	57	343	22	7	1

continued …

51	84	371	36	35	20
52	38	519	70	56	7
53	126	567	17	36	0
54	53	575	84	11	15
55	127	768	57	22	21
56	121	913	46	70	1
57	95	918	126	84	6
58	210	1139	39	45	35
59	91	1432	120	18	8
60	253	1485	127	57	1
61	174	1714	63	126	35
62	222	2200	210	120	28
63	331	2398	96	56	1
64	186	2632	166	40	21
65	463	3339	253	127	56
66	265	3830	102	210	9
67	475	4117	330	165	7
68	505	5053	223	74	70
69	408	6030	229	97	36
70	794	6515	463	253	2
71	451	7685	198	330	56
72	938	9369	496	221	84
73	770	10345	476	114	10
74	883	11802	331	224	28
75	1299	14422	793	463	126
76	859	16375	421	495	45
77	1732	18317	725	295	9
78	1221	22107	939	211	126
79	1821	25744	529	477	120
80	2069	28662	1289	793	12
81	1742	33909	897	716	84
82	3031	40166	1056	409	210
83	2080	45037	1732	435	55
84	3553	52226	950	940	37
85	3290	62273	2014	1288	252
86	3563	70781	1836	1011	165
87	5100	80888	1585	620	21
88	3822	96182	3021	912	210
89	6584	110947	1847	1733	330
90	5370	125925	3070	2004	67
91	7116	148408	3568	1420	121
92	8390	173220	2535	1055	462
93	7385	196706	5035	1852	220
94	11684	229296	3683	3021	58
95	9192	269402	4655	3015	462
96	13700	307653	6589	2040	495
97	13760	355221	4382	1967	88
98	14501	417810	8105	3585	331
99	20074	480873	7251	5025	792
100	16577	551927	7190	4435	287