2.3 Circles

- Note that "circumference", "diameter", "radius", etc. can refer either to the lines themselves ("things") or to the lengths of those lines; e.g., the length of a radius is often just called the radius.
- All circles are mathematically similar (like all squares, for example, but unlike, say, all right-angled triangles).
- You can write the area formula as $r^2\pi$ to avoid the danger of calculating $(\pi r)^2$ instead of πr^2.
- There are lots of definitions to grasp: an **arc** is part of the circumference of a circle; a **chord** is a straight line joining two points on the circumference (a diameter is a chord that goes through the centre); a **tangent** is a straight line touching the circumference at one point only; a **sector** is the area between an arc and two radii (a **semicircle** is a sector which is half a circle; a **quadrant** is a sector which is a quarter of a circle); a **segment** is the area between a chord and an arc. (Segments and sectors are easy to muddle up – a semicircle is both.) Circumference is just the perimeter of a circle.
- Material using Pythagoras' theorem in the context of circles is in section 2.7.

2.3.1 **NEED** string or tape measures and "round" objects or "Circles" sheet, callipers if you have them.
Practical Investigation: we're going to discover something interesting about circles.
Bring in or find circular objects (or objects with circular cross-section): dinner plate, clock, football, window, tiles, rubber, pencil sharpener, food tin, cup, marker pen, bin, sticky tape, someone's arm.
Measure the circumference and the diameter. Is there a connection between these two amounts?

Divide the circumference by the diameter (use the same units). What do you get?

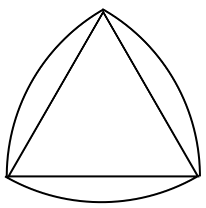
Pupils will realise that you can't get π very accurately by this method!

This leads to a value of π of about 3 (or "3 and a bit"). It's nice to demonstrate this "3 and a bit" if there's a fairly large (> 1 m diameter) circular object in school. Wrap the string around the outside and cut it the length of the circumference. Measure with it across the middle 3 times, and "a bit" is left over. (This can be quite a memorable demonstration.)

It's pretty amazing that $c = \pi d$ works regardless of scale; e.g., for a microscopic water drop or a giant star or planet's orbit.

In practice, it's usually easier to measure the diameter than the circumference, because straight lines are easier to measure accurately, but sometimes you can't "get at" the diameter (e.g., a pipe), and then it's useful to be able to calculate *the diameter from the circumference.*

2.3.2 What is it about circles that makes them good for wheels? Is there any other kind of shape that would do?



This shape is also used for drills that drill "square" holes (almost square – the corners aren't quite right).

Which other polygons can you make curvy versions of like this?

These are sometimes called "rolling polygons".

Answer: It's their constant "width" (diameter) regardless of orientation, so that whatever is travelling on top is always the same height off the ground.

Other shapes do that; e.g., a Reuleaux triangle (Franz Reuleaux, 1829-1905), formed by adding arcs to each side of an equilateral triangle (radius the same as the lengths of the sides of the triangle) – see left.
Although something resting on top would be carried horizontally, the centre *of the wheel wobbles up and down, so it wouldn't be any good on an axle.*

It works for all the regular polygons that have an odd *number of sides; seven-sided versions are used for 20 p and 50 p coins. Their constant width regardless of orientation helps in slot machines.*

2.3.3 Could you describe a circle over the telephone to someone who didn't know what one was? (Imagine

This is quite hard, although it seems like such a simple thing! You could say "a set of all the possible an alien who doesn't know, for example, what a football looks like or what we mean by "round".)

points that are a certain fixed distance from a fixed point in 2 dimensions."
(This would define a sphere in 3 dimensions.)

2.3.4 **NEED** sheets of circles drawn on 1 cm × 1 cm squared paper (containing circles of radius 6 cm/7 cm and 5 cm/8 cm).
We're looking for a connection between the radius of a circle and its area. Count 1 cm^2 squares (count the square if the circle covers half or more of the square, otherwise ignore it). Make a table of the radius versus area and look for a pattern.

Especially with the larger circles, it is sensible to mark off a big square of 1 cm^2 squares in the middle of the circle and find its area by multiplication, as that saves counting every single 1 cm^2. Then you can count the ones round the edge and add the two amounts.

You could divide up the work among the class so that, perhaps in groups of 2 or 3, pupils work on a couple of different-sized circles. The teacher can collect all the results on the board (doing some kind of average of the results people contribute, rejecting anything way out).
Someone could try the 3 cm, 4 cm and 9 cm radius circles as well.

Calculated results (typically you get within a couple of cm^2 experimentally):

radius (cm)	***area (cm^2)***
3	*28.2*
4	*50.2*
5	*78.5*
6	*113.1*
7	*153.9*
8	*201.1*
9	*254.5*

A clue to help with seeing the connection is to square the radius numbers and then look for a pattern. Make predictions and check.

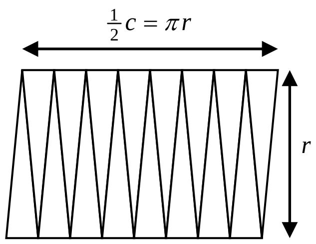
Pupils can plot the results on a graph (area on the vertical axis, radius on the horizontal).

Should get a parabola curve.

Using compasses and 1 cm × 1 cm squared paper, try to draw a circle with an area of exactly (or as near as you can) 100 cm^2. Use the graph to decide what the radius ought to be. Check by counting the squares.

Answer: the exact radius needed is $\sqrt{\frac{100}{\pi}} = 5.64$ *cm.*

2.3.5 **NEED** scissors, glue and 5 cm radius circles (draw with compasses). Cut out the circle and divide it roughly into sixteenths (8 lines), like cutting up a cake. Cut along all the lines so that you get 16 sectors of the circle.
Arrange them into an approximate "rectangle/parallelogram".



So the area is $\pi r \times r = \pi r^2$.

A circle this size produces a "rectangle" that fits nicely on an approximately A5 exercise book page.

This is more than an approximation, because we can imagine splitting up the circle into 32, 64, 128, etc. pieces; in fact, as many as we like, so we can make a shape which is as close to a rectangle/parallelogram as we like.

The more pieces we use, the more valid this argument becomes, so the area of the parallelogram gets closer and closer to the true area of the circle, so πr^2 *must be the true area of the circle.*

2.3.6 Function machines are useful for managing conversions between A, r, d and c.
It's easy to make up questions and put the values into a table.

See sheet.

2.3.7 The Number π is a *transcendental* number (it doesn't satisfy any polynomial equation with integer co-efficients). You can't write it as a fraction using integers (it's *irrational*). The decimal digits go on for

All transcendental numbers are irrational.
The opposite of transcendental is "algebraic".

Other transcendental numbers include e, e^{π}, $\ln 2$

ever and never go back to the beginning and repeat. Everyone's telephone number and credit card number is in there somewhere! *(It has not actually been proved that the digits of π are "random" in the sense that every possible combination of digits of a given length comes up equally often, but it is very probably true.)*

Ways of Calculating π:

1. $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots$, but this converges very slowly (145 terms to get 2 dp);
2. $\frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots$, which also converges slowly;
3. $\frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \ldots$, which converges quickly (only 3 terms to get 2 dp).
4. $\pi = 2 \times \frac{2}{1} \times \frac{2}{3} \times \frac{4}{3} \times \frac{4}{5} \times \frac{6}{5} \times \frac{6}{7} \times \ldots$, called Wallis' product (John Wallis, 1616-1703).

Some pupils may find these interesting:

$\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\,dx = \sqrt{2\pi}$ *and* $e^{i\pi} + 1 = 0$

and $\sqrt{2}^{\sqrt{2}}$. *No-one knows if* e^e, π^π *or* π^e *are transcendental.*

(See sheet for the first 10 000 or so digits of π *– you can photocopy back-to-back onto card and pass around the room: pupils may try to find their phone numbers!)*

2. is the Maclaurin (1698-1746) series for $\tan^{-1} 1$, *but it is also called the Leibniz (1646-1716) or Gregory (1638-1675) series.*

You could try these series on a spreadsheet. There are many other ways of calculating π.
To get a large number of digits, you need more efficient processes than these.

π *is a letter in the Greek alphabet (does anyone know Greek?). It has nothing to do with pies often being circular or pie charts or* P*ythagoras's Theorem!*

2.3.8 Imagine a cable lying flat on the ground all the way round the equator and back to where it started. If instead you wanted to support the cable all the way round on poles 10 m high, how much more cable would you need?
(We have to ignore the existence of the sea!)

(You don't need to know the radius of the earth, but it's 6.4 × 10^6 m, and you can provide it as unnecessary information if you like!)

An alternative version of this is the following puzzle: A businessman sets out on a journey, eventually returning to the place where he started. He claims that during his trip his head has travelled 12.6 m further than his feet have. How can that be possible?

Answer:
Additional cable = $2\pi(r+10) - 2\pi r$ *which is just* $2\pi 10 = 62.8$ *m.*
Much less than people generally expect.

The radius of the earth doesn't matter (it would be the same extra amount putting cable 10 m around a 2p piece), because for a larger circle you need a smaller proportion of a bigger amount; for a smaller circle a bigger proportion of a smaller amount.

He has been once round the equator and his height is 2 m.

2.3.9 Imagine a circular coin of radius r rolling round the edge of a square with perimeter p so that it never slips. How far does the centre of the coin move when the coin goes round the square once?

What if you rolled the coin round a different polygon (still with total perimeter p)?

What if you rolled it round an identical coin?

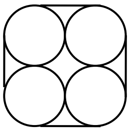
Answer: $p + 2\pi r$ *units. (It follows the edges of the polygon but also, at each vertex, the centre moves in an arc. By the time it gets back to the beginning it's turned through 360°, and that's where the extra* $2\pi r$ *comes from.)*

Same result. The polygon doesn't have to be regular, although it does need to be convex.

Effectively, the same result, with $p = 2\pi r$, *so the total is* $2\pi r + 2\pi r = 4\pi r$ *units.*
(The centre just moves round a circle with total radius $2r$, *so you can calculate* $2\pi(2r) = 4\pi r$, *the same answer.)*

2.3.10 Four large pipes, each of 1 m diameter, are held tightly together by a metal band as shown below. How long is the metal band?

Answer:
There are four quarter-circle arcs (one on each pipe) with a total length of $2\pi r = \pi$ *and four straight pieces with a total length of* $4 \times 2r = 4$, *so the total length of the metal band is*
$4 + \pi$ *metres = 7.14 m.*



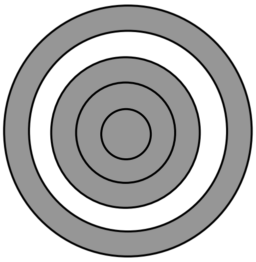
What if instead there are only three pipes?

What if there are n pipes?

By the same argument, length = $3+\pi$ m.

For n pipes, $n+\pi$ metres ($n>1$).
If $n=1$, it is just π metres.

2.3.11 (You can draw this reasonably well on a squared whiteboard.)
Which shaded area is bigger (could use different colours), the outer or the inner?

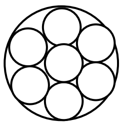


Answer:
Pupils may guess that they're the same, although many people think that the middle three rings look bigger.
Area of the outer ring = $\pi(5^2-4^2)$

= $\pi 3^2$ = area of first three rings.

If it were a "dartboard" it probably would be easier to hit the middle three rings than the outer one, because although the areas are the same the outer one has a very thin width.
(Imagine trying to hit a 4 cm × 4 cm square; that would be much easier than a 1 cm × 16 cm rectangle, although they have equal areas.)

2.3.12 Imagine a circular sheet of metal of diameter 6 m. What percentage of the metal will be wasted if you cut out two circles, each of diameter 3 m?
How many 2 m diameter circles can you cut out of the original 6 m diameter sheet? What would be the percentage wasted this time?



Answer:
$\frac{\textbf{area used}}{\textbf{total area}} = \frac{2\times\pi(1.5)^2}{\pi 3^2} = \frac{1}{2}$*, so 50% is wasted.*

Answer: 7 circles is the maximum (see drawing on the left)

With 2 m circles, $\frac{\textbf{area used}}{\textbf{total area}} = \frac{7\times\pi 1^2}{\pi 3^2} = \frac{7}{9}$*, so only* $\frac{2}{9}$ *or 22% is wasted now.*

2.3.13 A washing machine has a drum of diameter 50 cm and spins clothes at 1100 rpm (revolutions per minute). How far do a pair of trousers travel if they are spun for 5 minutes? (Assume they stick to the inside of the drum throughout.)
How fast are they going?

Answer: There's a total of 5 × 1100 = 5500 revolutions, each of which is a distance of πd = 0.5π = 1.6 m, so the total distance = 5500 × 1.6 = 8.6 km!
Speed = distance/time = 8.6/$\frac{1}{12}$ =
about 100 kph! (That's why its good if the door won't open until it's finished spinning!)

2.3.14 How many times do the wheels on a car go round when the car travels 1 mile?

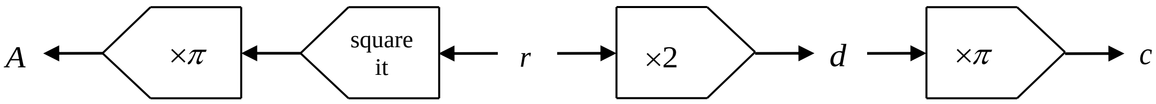
Assume a diameter of about 0.5 m.

Answer:
Circumference = πd = 0.5π = 1.6 m.
1 mile = 1.6 km, so number of rotations = 1600/1.6 = 1000 times. (This assumes that the wheel doesn't slip at all on the ground.)

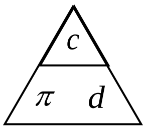
2.3.15 If a car has a turning circle (kerb-to-kerb) of 10 m, estimate the size of the narrowest road in which it could perform a three-point-turn.
(Turning circle means that on full lock at low speed the car could just follow a circle of this diameter; i.e., the car could just manage a U-turn in a street 10 m wide.)

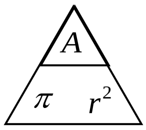
Answer: about half as much, 5 m in this case, because it can turn about 90° clockwise (viewed from above) before reaching the kerb and then reverse another 90° (still clockwise from above) before driving off.
(This assumes that the driver switches from right-lock to left-lock very quickly.)

Converting Area, Radius, Diameter and Circumference of Circles



Using the triangles below, cover up the variable that you want to find, and you can “see” the formula; e.g., $d = \frac{c}{\pi}$, etc.





Fill in the gaps in tables like this (choose where to leave out values). Vary the units.

r	***d***	***c***	***A***
6	12	37.70	113.10
1	2	6.28	3.14
25	50	157.08	1963.50
14	28	87.96	615.75
7.4	14.8	46.50	172.03
11	22	69.12	380.13
28	56	175.93	2463.01
35.8	71.6	224.94	4026.39
254	508	1595.93	202682.99
5	10	31.42	78.54
42	84	263.89	5541.77
0.75	1.5	4.71	1.77

Earth, Sun, Satellites

earth’s mean radius: 6.4×10^6 m mean distance from earth to sun: 1.5×10^{11} m
height above earth’s surface of geostationary satellites: 3.6×10^7 m

Use this data to answer these questions.

1. How far does someone standing on the equator move in 24 hours?
   First take account of the rotation of the earth.

   Answer: $2\pi r_{earth}$ = 40 000 km.

   Then think about the earth’s movement round the sun.

   Answer: $\frac{2\pi r_{earth-to-sun}}{365} = 2.6 \times 10^6$ km

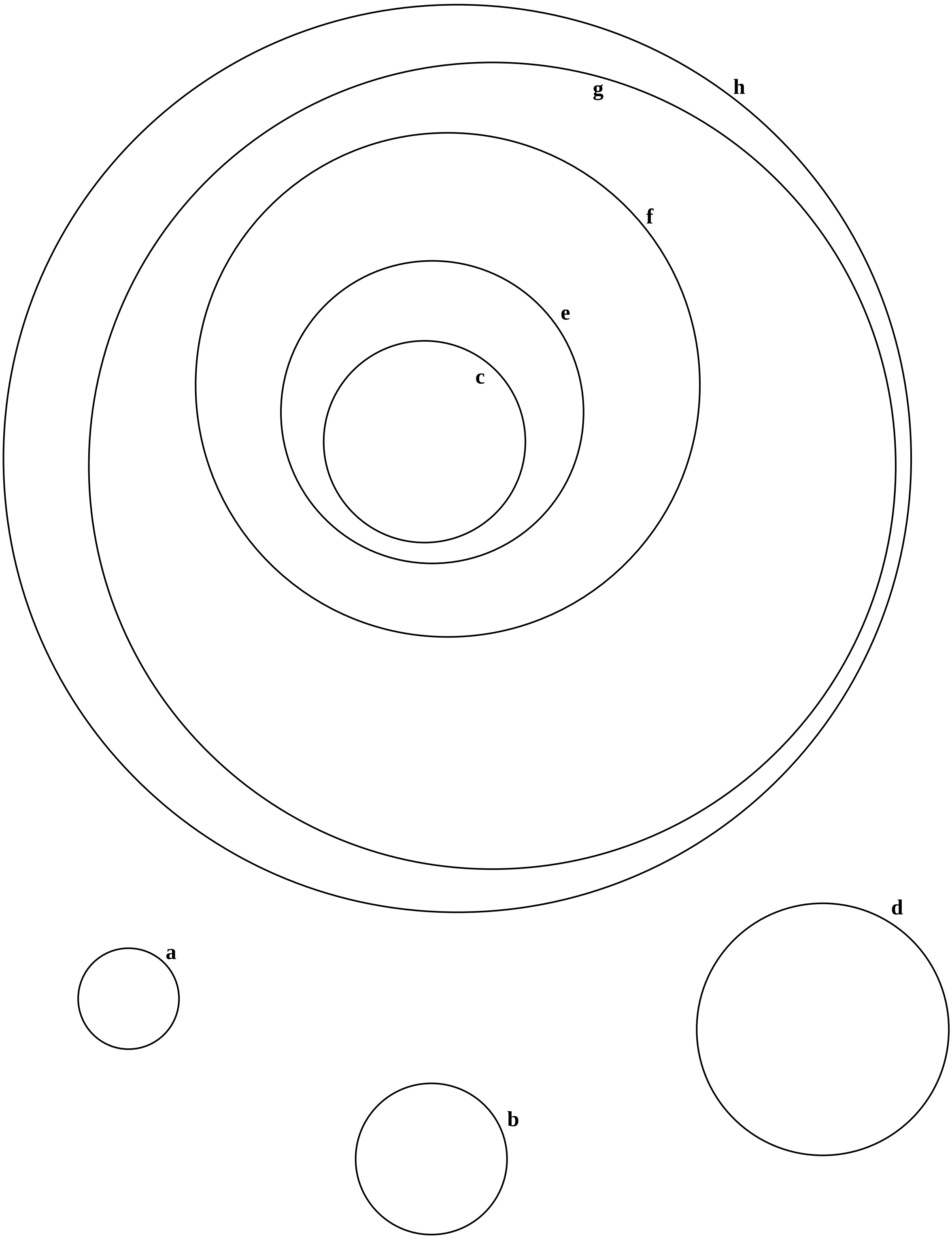
2. How fast does a geostationary satellite have to move in space?
   A geostationary satellite is one which is always in the same position above the surface of the earth as the earth rotates.

   Answer: $\frac{2\pi r_{earth-to-satellite}}{24}$ = 11 000 kph.

Circles

Measure the *circumference* (use string, a strip of paper or a tape measure) and the *diameter* of each circle.
Record your results in a table.
What do you notice?

Circumference: distance around the edge
Diameter: distance across the middle



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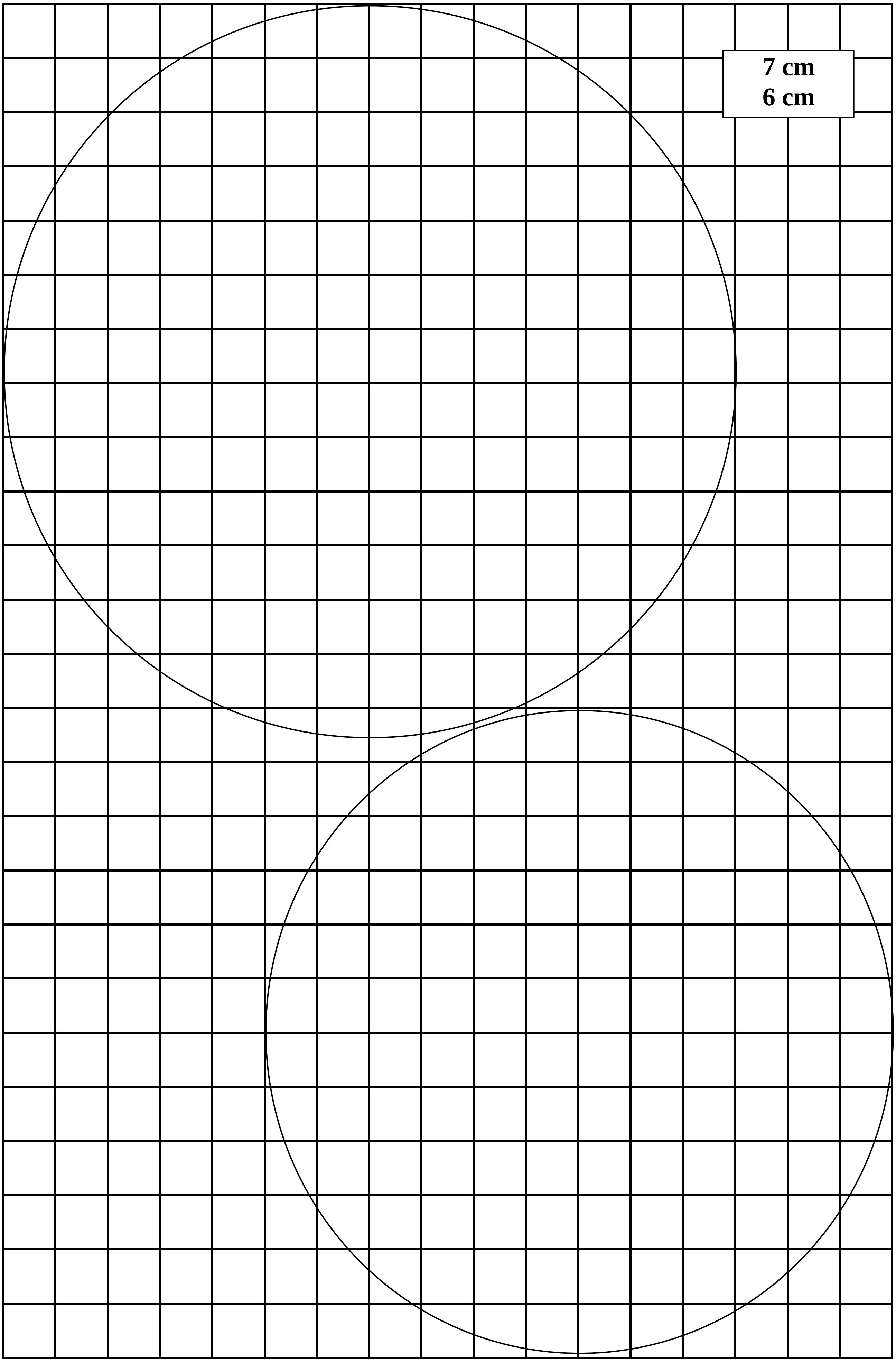
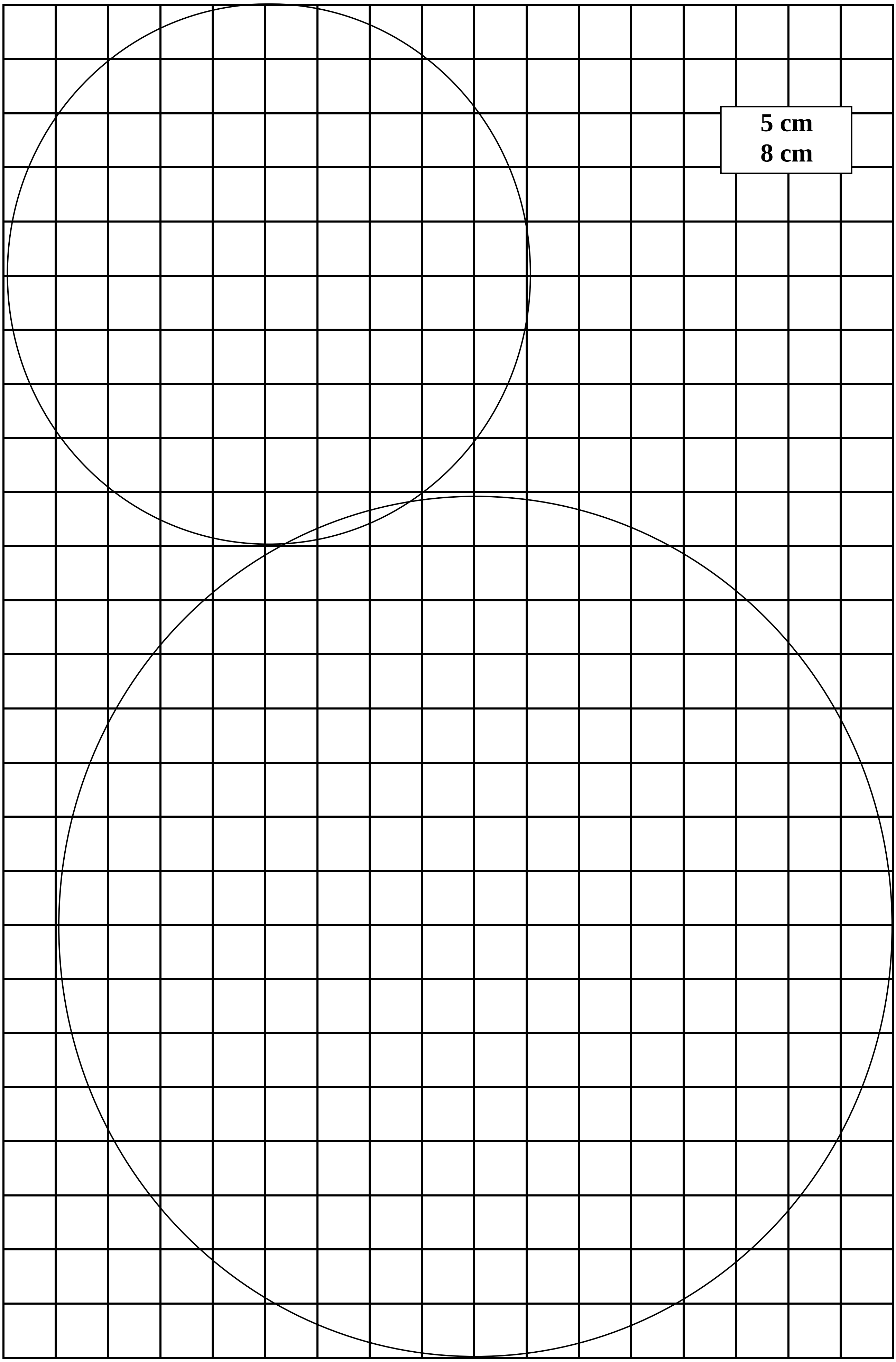

7 cm
6 cm


5 cm
8 cm