2.8 Loci and Constructions

- Strictly speaking, “constructions” can be done with pencil, compasses and straight-edge only. No measuring with ruler or protractor is allowed. But often this topic gets merged with scale drawing, so this distinction is lost.
- In drawing work it’s worth aiming for an accuracy of $\pm 1°$ and ± 1 mm. A sharp pencil helps.
- When using compasses, it can be useful to have a screwdriver handy for tightening them up.
- For some varied loci to draw, see the sheet.

2.8.1 **NEED** various props (e.g., clock, paper plate, door with handle, large book that will stand up on its end). (See sheet.)
Explain that locus means “all the possible positions that fit a particular rule”.
On scrap paper, pupils draw the loci for various situations described by the teacher.

Practical Lesson to introduce the concept.

You can consider that a point moves according to a rule, or if you prefer “points” to be “fixed” then the locus is the set of all points that satisfy a particular condition. It may be that one or other of these perspectives may be more helpful depending on the context of the particular locus problem.

2.8.2 **NEED** outdoor or large indoor space (playground, hall, gym, etc.) People maths.
Follow these instructions, then try to describe the loci.
It’s important to be able to describe precisely the lines that you get.

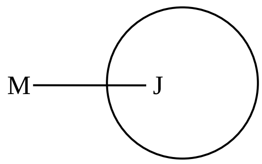
1. Leo, stay where you are. Everyone else, get 2 m away from Leo. (“Are you 2 m away from Leo? Could I stand here?”)
2. Max and Jo, stay where you are. Everyone else, stand somewhere where you are the same distance from Max as you are from Jo.
3. (Harder) Stand exactly twice as far from Max as you are from Jo. (Need to realise that there are points “in between” M and J and points beyond J but none beyond M.)
4. Get exactly 1 m away from that wall (a long wall).
5. Get exactly 1 m away from that wall/hedge (a shorter wall with ends).
6. Stand so that you’re the same distance from Jamie as you are from that wall.

There may be other possibilities according to the space and objects you have available. If you don’t have a suitable wall/hedge, you can use chalk to mark a line on the ground.

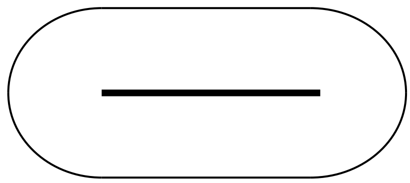
(If it isn’t practicable to do this with people, you could move coloured counters or cubes on a desk so as to satisfy these conditions.)

Should produce …

1. *a circle, radius 2 m, centred on Leo;*
2. *the perpendicular bisector of the line segment joining Max and Jo;*
3. *If stuck, the teacher can stand $\frac{2}{3}$ of the way from M to J and show that this point is acceptable or the same distance M is from J the other side of J (along MJ extended).*
   Answer: a circle of radius $\frac{2}{3}$ of the distance between M and J, as shown below;



4. *a line parallel to the wall, 1 m from it;*
5. *parallel lines along the sides and semicircles round the ends (see below);*



6. *a parabola.*

2.8.3 Think of some examples of loci in everyday life and describe them in words.

e.g., grass watered by garden sprinklers, school catchment areas, bomb blast radii, TV transmitter areas.

2.8.4 **NEED** compasses. The Goat and the Shed.
Using a scale of 1 cm to 1 m, do a scale drawing in

Answers:

1. *Three quarters of a circle of grass around the*

the middle of the page of a rectangular shed that is 3 m by 5 m.
Do a plan view (from above).
The shed is surrounded by grass and a goat is tied up to one outside corner of the shed.

1. If the rope is 2 m long, shade in the grass that the goat can eat.
   Calculate the area of grass it can eat.

2. The goat would like more freedom (and grass!), so the rope is replaced by one that is 4 m long. On a new drawing of the shed, again shade in the grass that the goat can reach and calculate its area.

3. What happens if the length of the rope is increased to 6 m?

4. How long does the rope have to be to reach all the way round the shed? How much grass can the goat eat then?

5. (Much harder) Finally a 10 m rope is tried. How much grass can the goat eat now?

What assumptions have we made in solving these problems?
We've i*gnored the difference between the place where the rope ends and the "reach" of the goat's mouth; assumed the rope is always horizontal and the walls of the shed are vertical;*
assumed the rope doesn't stretch or break or get chewed through!; assumed no limits to the area beyond the shed; etc.

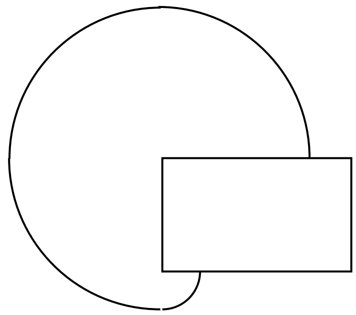
2.8.5 Cats and Dogs.
A cat and a dog don't like each other. They are kept on leads fastened to a wall at separate points. If the dog's lead is 2 m long, draw diagrams to find out where the cat's lead could be fastened and how long it could be so that the two animals would never be able to reach each other.
Try different shaped areas.
Use a scale of 1 cm to 1 m.

2.8.6 Ellipse.
Draw a circle with radius 5 cm and mark a point inside the circle about 4 cm from the centre. Fold the circle so that a point on the circumference just touches the marked point. Unfold the paper and draw a line along the fold mark. Repeat for a different point on the circumference. (Keep the point inside the circle the same.)

2.8.7 John wants to walk from his house H to school S *via* the river (to feed the ducks) along as *short* a total distance as possible.
What path should he take? (He can feed the ducks just as well at any point along the river.)

shed.
Area = $\frac{3}{4}\pi 2^2 = 9.42$ *m²*.

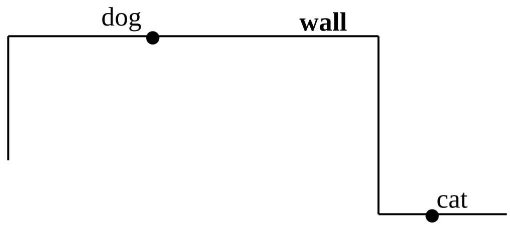
2. *Along the 5 m side the situation is similar to before, but along the 3 m side the rope will reach past the next corner of the shed and will catch and the goat can then make an additional quadrant of radius 1 m (see below).*



Area = $\frac{3}{4}\pi 4^2 + \frac{1}{4}\pi 1^2 = 38.48$ m^2.
3. *The rope catches at both corners now, so area =* $\frac{3}{4}\pi 6^2 + \frac{1}{4}\pi 1^2 + \frac{1}{4}\pi 3^2 = 92.68$ m^2.
4. *8 m to reach right round the shed, and now area =* $\frac{3}{4}\pi 8^2 + \frac{1}{4}\pi 3^2 + \frac{1}{4}\pi 5^2 = 177.50$ m^2.
5. *Now the same bit of grass can be reached by the goat going clockwise or anticlockwise around the shed, so this method of calculation will overestimate the area.*
   The true answer is about 280 m^2.

Some of the principles of mathematical modelling again.

e.g.,

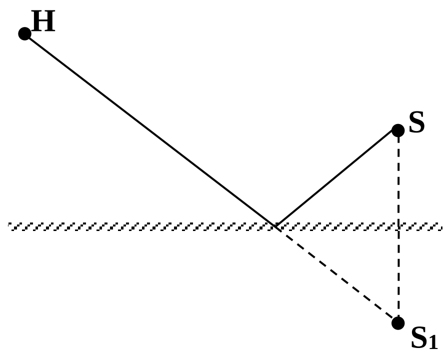


Another method for generating an ellipse is to tie string slack between two fixed points and push the string taut with a pencil. Then draw with the pencil keeping the string tight. The two fixed points are the foci *of the ellipse.*
(See "Loci" sheet.)

Eventually you will end up with an ellipse.

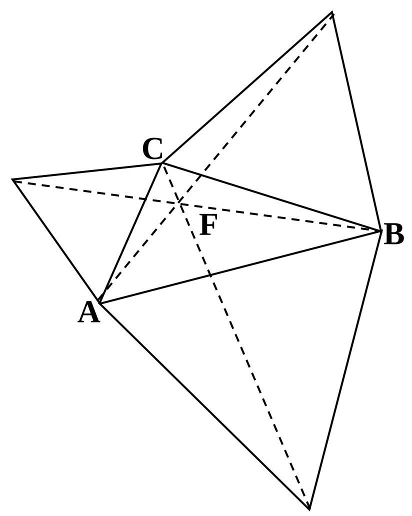
Answer:

You can fix H and S at certain perpendicular distances from the river, and a certain distance apart, and try by trial and improvement to get the shortest total distance.
(e.g., scale 1 cm = 100 m)



Light travels via *the shortest route between two points (this is what we mean by "straight"), so imagine that the river is a mirror and H is a light source. To get the reflected beam to go through S, construct S_1, the reflection of S in the mirror and join H and S_1 with a straight line. Where this intersects the mirror is the point on the river that John should walk to.*

2.8.8 There are three towns A, B and C at different distances from each other. Where should you build roads to connect the three towns so that the minimum length of road is constructed?
Do a clear drawing (A, B and C can be the vertices of any scalene triangle) and mark on the exact positions of the roads.



Answer:

Two possibilities:

1. *If one of the angles in the triangle is ≥ 120°, then road should be built from this vertex to the other two;*
2. *If none of the angles is as big as 120°, then you need to find the "Fermat Point", which when joined to each vertex makes three lines all at 120° to each other. One way of finding this point is to construct equilateral triangles on each side of the original triangle and join the outermost vertices of these to the opposite vertices of the original triangle. Where the lines cross is the Fermat Point. See diagram on the left.*

The Fermat point is F.
The roads should be built along AF, BF and CF.

2.8.9 **NEED** A4 plain paper and sticky tape.
Do an accurate scale drawing of a football pitch using this information.

1. The pitch should be 100 yards long and 50 yards wide.
2. The centre circle should have a radius of 10 yards.
3. The goal is 8 yards wide and surrounded by the 6-yard box (a rectangle 20 yards by 6 yards).
4. The penalty area is the 18-yard box (a rectangle 44 yards by 18 yards).

The penalty point should be 12 yards from the goal line and half way across the width of the pitch.

You could use three sheets of A4 paper taped together and a scale of 1 cm for every 2 yards.

Green paper adds a bit of realism!

Pupils could of course research the measurements for other kinds of pitches and draw those if they prefer.

2.8.10 Drawing accurate triangles.
Use compasses to draw these triangles as accurately as possible.

1. AB = 10 cm; AC = 8 cm; BC = 8 cm;

Answers:

1. *isosceles: A = B = 51.3°; C = 77.4°;*
2. *r-angled: A = 36.9°; B = 53.1°; C = 90.0°;*
3. *obt-ang: A = 49.5°; B = 22.3°; C = 108.2°;*

2. AB = 10 cm; AC = 8 cm; BC = 6 cm;
3. AB = 10 cm; AC = 4 cm; BC = 8 cm;
4. AB = 10 cm; AC = 4 cm; BC = 5 cm.

Pupils can check the accuracy of their drawings by measuring the angles.
Exact values are given on the right.

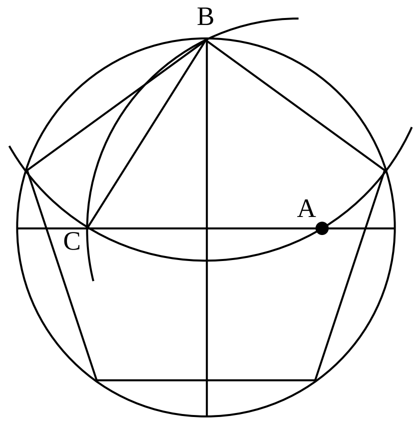
4. no such triangle because in any triangle the sum of the two shorter sides must be more than the longest side (the shortest distance from A to B must be the line AB, so AC and BC together must come to more than this, otherwise it won't join up).

2.8.11 Drawing Polygons with Compasses.
Pupils first need to train their compasses to behave properly, so it's worth starting by making sure everyone can draw a circle and get a single clean smooth line (no wobbles).

This can make good display work.

1. Regular Hexagon – the easiest.
Draw a circle of radius 5 cm in the middle of the page. Keep the compasses open at 5 cm. Put a mark on the circumference at the top of the circle. Put the compass point here and mark off the two points the pencil reaches to on the circle. Repeat at those points until you have 6 equally spaced points round the circle. Join them up with a ruler.

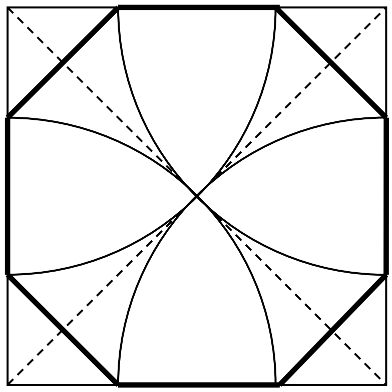
2. Regular Pentagon – need to be good at following instructions! (See diagram right.)
Draw a circle of radius 5 cm in the middle of the page. Draw two perpendicular diameters (horizontal and vertical).
Set the compasses to a radius of 2.5 cm. With centre A (mid-point of horizontal radius, see right) and radius AB draw an arc that cuts the circumference at B and the horizontal diameter at C.
Then draw an arc centred at B with radius BC to cut the circumference twice. This distance (BC) is the length of the side of the regular pentagon, so step this round the circumference and join up the points.



To draw the perpendicular diameters, pupils could use the grid lines if using squared paper; otherwise need to do a perpendicular bisector.

Could bisect the radius to get the 2.5 cm, or "cheat" by measuring with a ruler!

3. Regular Octagon – not too hard.
Draw a square with sides 8 cm and draw in the diagonals. Set the compasses to a radius of half the diagonal (the distance from a vertex to the centre of the square) and draw an arc centred on one of the vertices so that it crosses the two adjacent sides. Repeat for the other three vertices and join up the 8 crossing points with a ruler (see right).



2.8.12 Constructions with compasses:
1. Perpendicular Bisector of a line segment:
   all the points that are the same distance from one end of the given line as they are from the other;
2. Angle Bisector of two non-parallel lines:
   the line that makes the same angle with each of the starting lines.

This makes use of the fact that the diagonals of a rhombus bisect each other.
It doesn't matter how far the compasses are opened so long as it's more than half the distance between the two starting points and it doesn't alter.

Loci

Try to sketch these loci as carefully as you can.

1 The locus of a point on the front tip of an aeroplane as it comes in to land.

2 The locus of a white dot of paint on the moving end of the minute hand on a clock.
What if the dot of paint is only half way along the minute hand?

3 The locus, viewed from above, of a point on the door handle when I open the door.

4 The locus of a ball thrown over a wall.

5 The locus of a point on the rim of a bicycle wheel as the bicycle moves without slipping along a flat horizontal road at a steady speed.
What would happen if the bicycle speeded up?

6 The locus of a point on the flange of a train wheel as the train moves along a flat horizontal track at a steady speed.

7 The locus of a point part-way along one of the spokes of a bicycle wheel as the bicycle moves along a flat horizontal road at a steady speed.

8 The locus of a point on the circumference of a circle as it rolls (without slipping) around another circle that is the same size.
e.g., try rolling a 10 p coin around another one.

9 The locus of a point whose total distance from two fixed points is a constant.
e.g., tie a slack piece of string between the two points and use a pencil to make the string taut – draw with the pencil, keeping the string taut.

10 The locus of the points all the way along a uniform (same all the way along) rope suspended between two horizontal points high enough so the rope doesn't touch the ground.

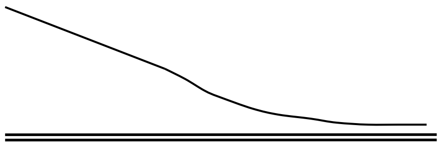
11 The locus of a point on the top right corner of a book as the book "rolls" (without slipping) along a table.
Draw this one as accurately as you can. You could take the book as being 15 cm wide by 20 cm high (i.e., a 3:4 ratio of width to height). Its thickness doesn't matter so long as it is thick enough to stand up without falling over.

12 The locus of a point mid-way along a ladder as the ladder slides down from a vertical position against a wall until it is horizontal. (The top of the ladder slides down the wall; the bottom of the ladder slides along the ground.)

What other loci can you think of and sketch?

Loci ANSWERS

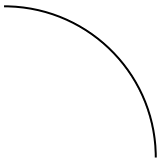
1 Aeroplane landing: something like this



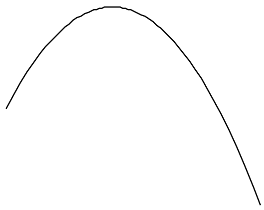
2 Dot on the end of the minute hand of a clock: a circle of radius equal to the length of the minute hand and centred on the middle of the clock.

If the dot were only half way along, the circle would have the same centre but half the radius.

3 Door handle as the door opens:
roughly a quarter of a circle.
(Some pupils will think it's a straight line.)



4 Ball thrown over a wall:



The curve is part of a *parabola*
The equation is $y = ax^2 + b$, $a > 0$.

5 Bicycle wheel: point on the rim

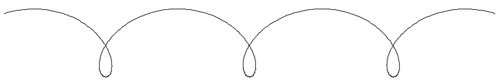
(You can demonstrate this with a plate marked with a dot and rolled along a table – when the dot touches the table it doesn't slide backwards, so there are no loops.)

Called a **cycloid**. Looks a bit like a row of semicircles but isn't.
If the bicycle speeded up, it would make no difference to the curve, although the later parts would get drawn faster.

The cycloid has lots of interesting properties; e.g.,
- it's the strongest shape for the arch of a bridge;
- the area under each "hump" is 3 times the area of the bicycle wheel;
- if you turn the shape upside down and roll a marble down the inside it takes the same amount of time to reach the bottom wherever you start it from – it's also the solution to the "brachistochrone problem": what path should a particle roll down to get from one point to a lower point in the shortest possible time?

The parametric equations are $x = r(\theta - \sin\theta)$ and $y = r(1 - \cos\theta)$, where $r > 0$.

6 Train wheel: point on the flange



Called a **prolate cycloid**.

This is the solution to the following puzzle: If a train is travelling from London to Edinburgh, what points on the train are (at a given instant) moving towards London?
(Assume that the train keeps going throughout the journey.)

Answer: points on the flanges of the wheels. (Passengers walking down the train will still be heading towards Edinburgh because their speed relative to the carriage will be tiny compared with the speed of the train.)

The parametric equations are $x = r\theta - d\sin\theta$ and $y = r - d\cos\theta$, where $d > r$.

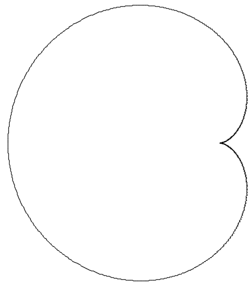
Loci (continued) ANSWERS

7 Bicycle wheel: point part-way along one of the spokes

Called a **curtate cycloid**.

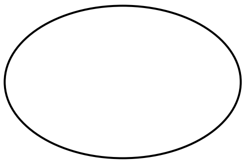
The parametric equations are $x = r\theta - d\sin\theta$ and $y = r - d\cos\theta$, where $0 < d < r$.

8 A point on the circumference of a coin rolling around another coin:



Called a **cardioid** (means "heart-shaped", like "cardiac").
The polar equation is $r = 2a(1 \pm \cos\theta)$.

9 Pencil and string: an **ellipse**.

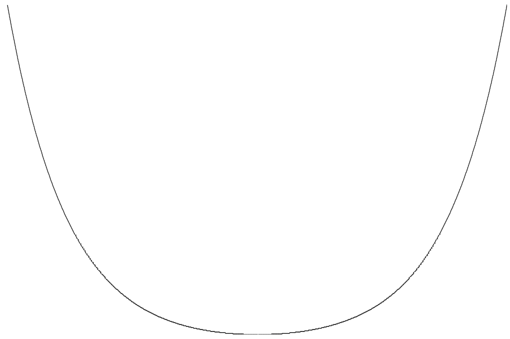


The parametric equations are $x = a\cos\theta$ and $y = b\sin\theta$,

and the Cartesian equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

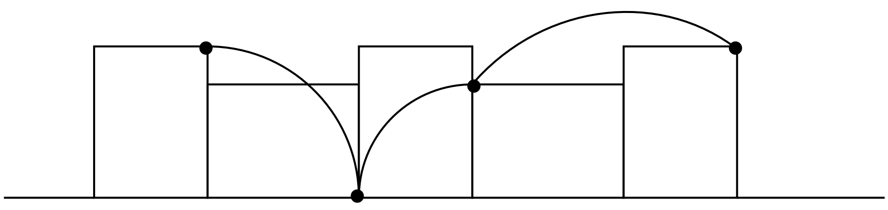
An alternative definition of an ellipse is "the locus of a point which moves so that the ratio of its distance from a fixed point to its perpendicular distance from a fixed line is a constant <1". The fixed point is called the *focus* and the fixed line the *directrix*.

10 Rope suspended between two points:



Called a **catenary**.
Looks a bit like a parabola but it isn't.
The equation is $y = a\cosh\left(\frac{x}{a}\right)$.

11 Corner of book balanced on its end as it "rolls" along the table:
(It helps to draw in the positions of the book at each stage.)



The book is moving from left to right. After it rotates 90°, the next rotation is about the dot, so this point doesn't move. Notice that in the final 90° rotation the dot moves *above* the height of its final position.

The radii of the arcs are 20 cm, 15 cm and $\sqrt{15^2 + 20^2} = 25$ cm (Pythagoras' Theorem, a 5 times enlargement in cm of a 3-4-5 triangle).

12 Mid-point of ladder as it slides (not tips) from against a vertical wall until it is horizontal.
The locus is a quarter of the circumference of a circle with radius half the length of the ladder.

Triangle Properties and Words

name	definition	properties
incentre	point where the 3 ***angle bisectors*** intersect	the ***incentre*** is the centre of the ***inscribed circle***, which touches each of the sides of the triangle
orthocentre	point where the 3 ***altitudes*** intersect (an ***altitude*** is the line joining a vertex to the opposite side so that it is perpendicular to the opposite side)	if you join together the feet of the ***altitudes***, they make another triangle called the ***pedal triangle***, and the ***orthocentre*** is the ***incentre*** of this ***pedal triangle***
circumcentre	point where the 3 ***perpendicular bisectors*** of the triangle intersect	the ***circumcentre*** is the centre of the ***circumscribed circle***, which goes through all 3 vertices
centroid	point where the 3 ***medians*** intersect (a ***median*** is the line joining a vertex to the mid-point of the opposite side)	if the triangle were a thin uniform lamina, the ***centroid*** would be the position of the centre of mass; the ***centroid*** divides the ***medians*** in the ratio 1:2

The orthocentre, circumcentre and centroid are collinear (Euler's line).

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