2.9 3-D Solids and Nets

- You need to decide whether or not to use the words “shape” (2-d) and “solid” (3-d) interchangeably. It can be helpful to avoid saying “shape” when referring to a 3-d object.
  Pupils will say “square” when they mean “cube”, and you can say, “a square is the *shape* on the *end* of a cube, but what’s the whole *solid* called?” (Note that in common usage, “solid” means “hard”, so that a pile of sand or a sponge might not be considered solid (or not very), although in science they would be. Also liquids and gases are “solids” in maths!)
- Collect Easter egg boxes: often get isosceles trapezoidal prisms and the occasional pyramid (sometimes truncated). Easter holiday homework can be to look for unusual boxes and bring them in to be named! At other times of the year, chocolate boxes are often interesting solids.
- This can be an encouraging topic for some pupils who often find maths hard, because it relies on quite different skills (e.g., spatial awareness) from those needed in some other areas of maths.

2.9.1 Naming Solids. It’s very useful to have actual 3-d solids (cardboard boxes or plastic solids) to pass around the room. “What has David got? Where do you come across triangular prisms?”, etc.
Prism: in a certain direction parallel slices are all congruent (e.g., slices of bread) (same cross-section all the way through);
Pyramid: triangular faces that all meet at one point.

(See sheet of drawings, suitable for acetate: point and name: “Can anyone name them all?” Can turn the acetate round and over to vary the appearance.)

A cuboid has 6 rectangular faces: none need be square, or two opposite ones could be square or all 6 could be square, in which case it’s a cube.
Cubes, cuboids and cylinders are all prisms.
A triangular prism is a “tent” shape, and a typical glass or Perspex prism in Science will be a triangular prism.
Pencils are sometimes hexagonal prisms and sometimes cylinders (with cones at the point).
A tetrahedron is a triangle-based pyramid. The Egyptian pyramids are square-based pyramids.

“Hold up your solid if you think it’s a prism”, etc.

2.9.2 I-spy a solid in the classroom; e.g., “I can see a triangular prism” and others have to guess what the object is. Initially give no indication of size. You can also describe mathematically an object (perhaps on the school site) that everyone knows and others have to guess what it is.

e.g., “The object is a hollow cylinder of diameter 8 cm and length 2 cm.” Answer: a roll of sticky tape.

2.9.3 Prisms. Instead of just cataloguing solids as prisms or not prisms, you can do it the other way round by asking what solids these could be:
(All the solids are common ones.)

	“cross-sectional shape”	**prism?**
1	circle	yes
2	circle	no
3	triangle	yes
4	triangle	no
5	square	yes
6	square	no

Answers:
1. *cylinder*
2. *cone or sphere*
3. *triangular prism*
4. *tetrahedron*
5. *cube or cuboid*
6. *square-based pyramid or octahedron*

There are many other possible answers.

2.9.4 What very common everyday object has approximately these dimensions?
20 cm × 10 cm × 8 cm

Answer: ordinary house brick

Could estimate how many used in a house.

2.9.5 Polyhedra. A regular polyhedron (called Platonic) has the same regular polygon for all of its faces, and all its vertices are identical.
Find out how many regular polyhedra there are and what they are.

Answers: There are only 5:
- ***cube*** *(6 square faces),*
- ***regular tetrahedron*** *(4 equilateral triangle faces),*
- ***regular octahedron*** *(8 equilateral triangle faces),*

Named after Plato (427-347 BC).

- ***regular dodecahedron*** *(12 regular pentagon faces) and*
- ***regular icosahedron*** *(20 equilateral triangle faces).*

The regular tetrahedron, the regular octahedron and the regular icosahedron are ***deltahedra*** *(polyhedra whose faces are all equilateral triangles).*

NEED 3-d models or 2-d sketches of regular polyhedra. What is the connection between the numbers of vertices, faces and edges that they have?

Euler's formula (1707-1783):
vertices + faces = edges +2

Euler's relationship works for convex polyhedra with straight edges and flat faces so long as they don't have "holes" in them!

2.9.6 What symmetry does a cube have?

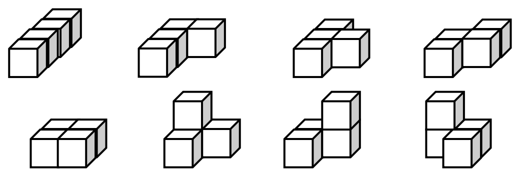
Answer: 9 planes of symmetry (3 parallel to faces and 6 at 45° to pairs of faces); 13 axes of symmetry (3 through the centre of opposite faces, 4 through opposite vertices and 6 through the mid-points of opposite edges).

2.9.7 **NEED** molecular model kit (Science dept.).
Chemical molecules and crystals often have symmetrical structures. You can imagine joining every atom to every other atom.
For example, trigonal planar (e.g., BF_3); octagonal (e.g., SF_6); tetrahedral (e.g., CH_4); square-based pyramidal (e.g., IF_5); trigonal bipyramidal (e.g., PF_5); etc.

Some of these molecules have these structures only approximately, and the atoms are always moving around anyway.

2.9.8 **NEED** isometric paper (see sheet) and interlocking cubes. Polycubes.
How many solids can you make by linking together four cubes?
Draw them on isometric paper.

"Isometric" means "equal distance"; the distances from any dot to its six nearest neighbours are all the same.
Must have the paper "portrait" so that there are vertical lines of dots.
The 8 tetracubes are shown below.



Which ones would look different in a mirror (ignore the colours of the cubes)?

Only the last two are "chiral" (have non-superimposable mirror images – like left and right hands). In fact, they are mirror images of each other. All the others have at least one plane of symmetry.

Chirality is important with chemical molecules. Different mirror image molecules (enantiomers) have different properties.

What if I give you another cube so that you have five?

There are 29 so-called "pentacubes", and with more cubes you quickly get vast numbers of polycubes (6 give 166, 7 give 1023, etc.).

2.9.9 **NEED** drawings of "impossible solids" – drawing these on isometric paper (perhaps enlarging them at the same time) is an interesting way of getting used to 3-d isometric drawing.

Suitable for display.

2.9.10 **NEED** "Cube or Not?" sheets, scissors, glue.
Pupils should be encouraged to find other nets that will make cubes and to make up puzzles for each other.

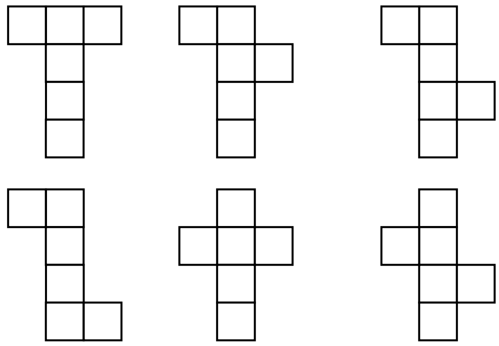
Answers:
A, C, E make cubes; B, D, F don't. ("ACE" is easy to remember as you discuss with pupils.)

How many possible nets are there for a cube?

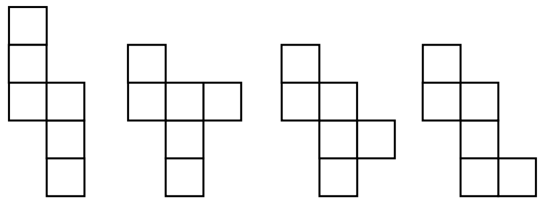
Out of 35 possible hexominoes, only 11 are nets for a cube. Counting systematically:
1. Four squares in a line (6 of these):

Here we count as the same nets which are rotations or reflections (turn it over) of each other.

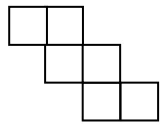
This is a good task to encourage systematic work.



2. Only three squares in a line (4 of these):



3. Only two squares in a line (just 1 of these):

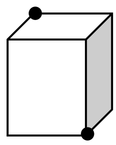


2.9.11 Flaps on a net.
Often we miss them out to keep things simple. If you need flaps for the glue, where do you have to put them?

(See sheets for cube, tetrahedron, triangular prism, octahedron, icosahedron and dodecahedron.)

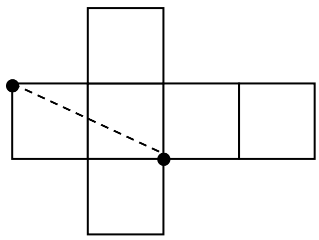
It generally works if you go clockwise or anticlockwise around the perimeter of the net putting a flap on every other edge that you come to.
(If in doubt, it's better to put one, because you can always cut it off if you don't need it but it's more of a problem if you don't have one that you do turn out to need! Cutting off unnecessary flaps is a bit amateurish, though!)

2.9.12 (Need Pythagoras' Theorem)
A spider wants to crawl from the top left back corner of a cube room to the bottom right front corner. If the sides of the room are all 3 m, then what is the shortest distance the spider can crawl?
(No jumping/webbing, etc. allowed!)



You can make up similar puzzles using cuboid rooms, or more complicated solids. The spider can start in the middle of a wall.

Answer: Most people suggest going vertically down the wall and then diagonally across the floor, with a total of $3(1+\sqrt{2}) = 7.24$ *m, but there is a shorter way, most easily seen by drawing the net of the room:*



Here the distance is only $3\sqrt{5} = 6.71$ *m.*

2.9.13 What kind of paper would be the easiest to use to draw a net for these solids?

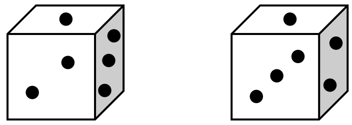
1. a cube or a cuboid;
2. a right-angled triangular prism;
3. a non-right-angled triangular prism;
4. a square-based pyramid;
5. a hexagonal-based pyramid;
6. a tetrahedron;
7. a cone;

Answers:
1. *squared (or square dotty) paper;*
2. *if you use a Pythagorean Triple (e.g., 3-4-5), then squared paper is easiest; otherwise, squared paper or plain paper and compasses is best;*
3. *plain paper and compasses or isometric paper if you're careful;*
4. *squared paper;*
5. *isometric paper or plain paper and compasses;*

8. a sphere.

6. isometric paper;
7. plain paper and compasses;
8. no net for a sphere is possible!

2.9.14 A normal (cube) dice has the numbers 1 to 6 on its six faces and the numbers on every pair of opposite faces add up to 7. Draw a net for such a dice. Are all such dice the same as each other?



Answer: There are two different possible dice like this, sometimes called left-handed and right-handed dice because they are mirror images of each other.

2.9.15 **NEED** acetate of two nets of a cube.
Ask questions like these:

When the net is folded up to make a cube,
1. which faces will be next to the number 1?
2. which face will be opposite the 5?
3. which face will the arrow point to?, etc.

Answers:
1. 2, 4, 6;
2. 3;
3. star.

2.9.16 **NEED** card, scissors, glue, sticky tape.
Shape Sorter.
This is an object small children play with to get used to matching different shapes/solids.
Each solid must fit through one hole only (otherwise the child will get confused!), yet it must go through the right hole reasonably easily (or the child will get frustrated!).

Design and make a shape sorter out of cardboard.
Pupils could use a ready-made box (e.g., a shoe box or cereal box) and just cut holes in it and make solids to fit through.
Ideally, when all the solids are inside the box the lid will go on for convenient storage.

A challenge is to make a shape sorter that uses only cubes and cuboids.

You cannot have more than one cylinder in your shape sorter. Why not?

The fact that circles have "infinite" rotational symmetry may be why manhole covers are round – there's no risk of the cover falling down the hole. (For example, hexagonal covers on hexagonal holes could do that.)

Pupils tend to be too adventurous at the beginning and need to be encouraged to "practise" by starting with cubes and cuboids before making a twelve-pointed star!

The "holes" need designing first to make sure that none of the solids will go through the wrong holes; e.g., you could begin with a cylinder of diameter 4 cm. A square that won't go through its hole has to have sides longer than $2\sqrt{2} = 2.8$ cm, but the sides of the square have to be less than 4 cm, otherwise the cylinder will go through its hole.
At each stage you have to check that nothing you've designed so far will go through the hole for anything else.

If the shapes are prisms, you can neglect the "length" dimension so long as it is long enough not to let the solid go through any of the holes.

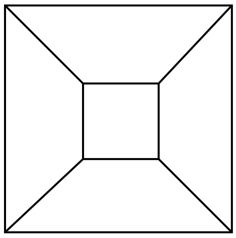
Because mathematically similar shapes (all circles are similar) of different sizes will always go through each other (so you can't have more than one cube, either, because all squares are similar).

2.9.17 How many colours do you need to colour the faces of a cube if no two faces that share an edge are allowed to have the same colour?
How many different ways can it be done if you have more than this many colours?

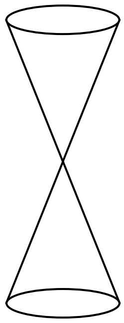
For these purposes, a cube is equivalent to the following 2-d "graph" (the "outside" corresponds to a face as well).

Answer: 3 is the minimum number of colours needed (colour opposite faces the same colour).

no. of colours	***no. of ways***
1	*0*
2	*0*
3	*1*
6	*30*



2.9.18 Conic Sections.
Imagine a double-cone shape, as below, that goes on forever in the up and down directions.

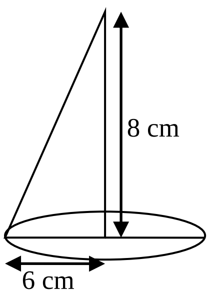


Describe the curves you would get if you sliced through it at different angles with a flat surface (plane).

Answers: Good visualisation exercise.

1. *horizontal plane: a circle (or dot if through the point);*
2. *plane at an angle less than steepness of the sides of the cone: ellipse (or dot if through the point);*
3. *plane at an angle equal to the steepness of the sides of the cone: parabola (or a straight line if through the point);*
4. *plane at an angle steeper than the sides of the cone: hyperbola – two separate bits of curve, discontinuous with asymptotes (or a pair of straight lines if through the point).*

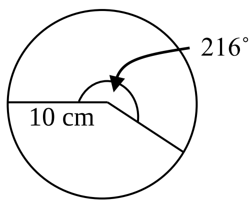
2.9.19 **NEED** A4 plain paper, scissors, sticky tape.
Net of a Cone.
I want to make a cone with a vertical height of 8 cm and a base radius of 6 cm.
What exactly will the net have to be?



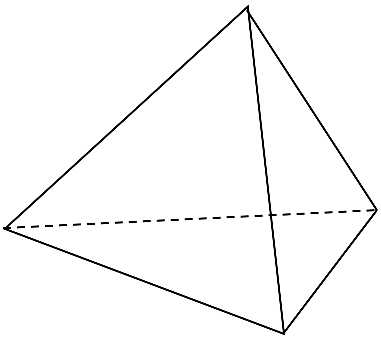
(The cone could be to hold chips or popcorn.)

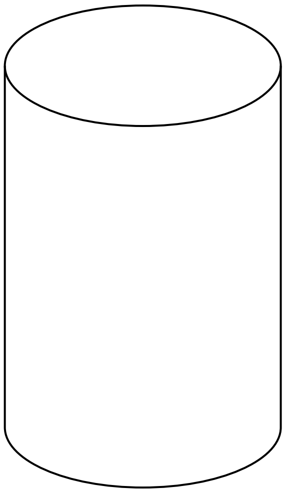
The cone also needs a circular base of radius 6 cm.
Can you cut out everything you need from one sheet of A4 paper?
Yes, if you're careful.

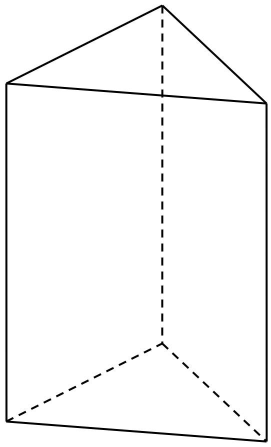
Answer:
You need to cut out a sector of a circle of radius equal to the slant height of the cone. By Pythagoras' Theorem, the sloping sides of the cone will go up $\sqrt{6^2+8^2}=10$ *cm (twice a 3-4-5 triangle). The circumference of the base of the cone will be* $2\pi r = 12\pi$*, so we need the sector of our 10 cm circle to have arc length* 12π*, and this will be* $\frac{12\pi}{20\pi}=\frac{3}{5}$ *of the 10 cm circle's circumference, or* $\frac{3}{5}\times 360 = 216^\circ$*. When cut out and folded up, this will make the required cone.*

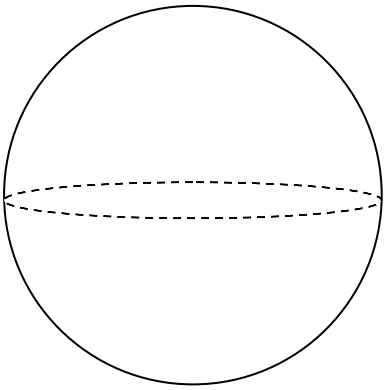


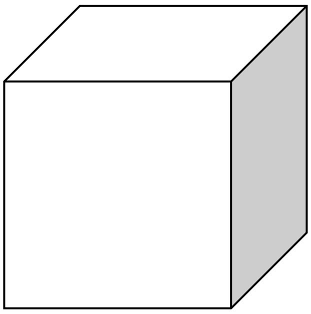
Volume $= \frac{1}{3}\pi r^2 h = 302$ *cm*^{3}.

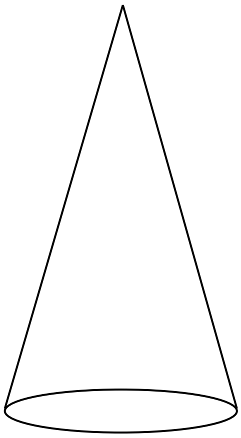


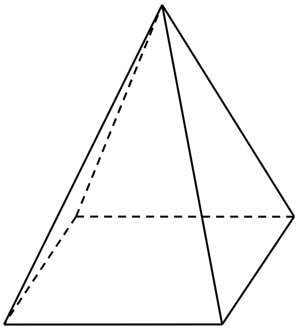


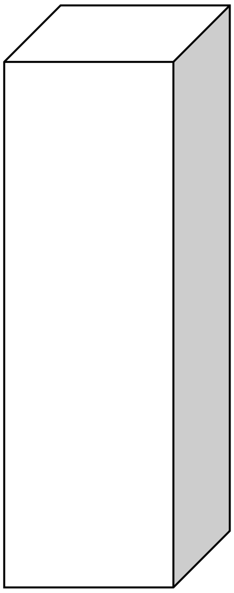


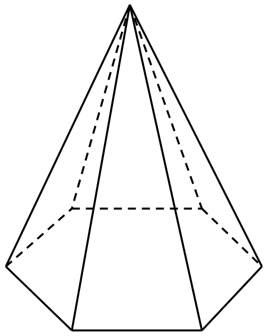






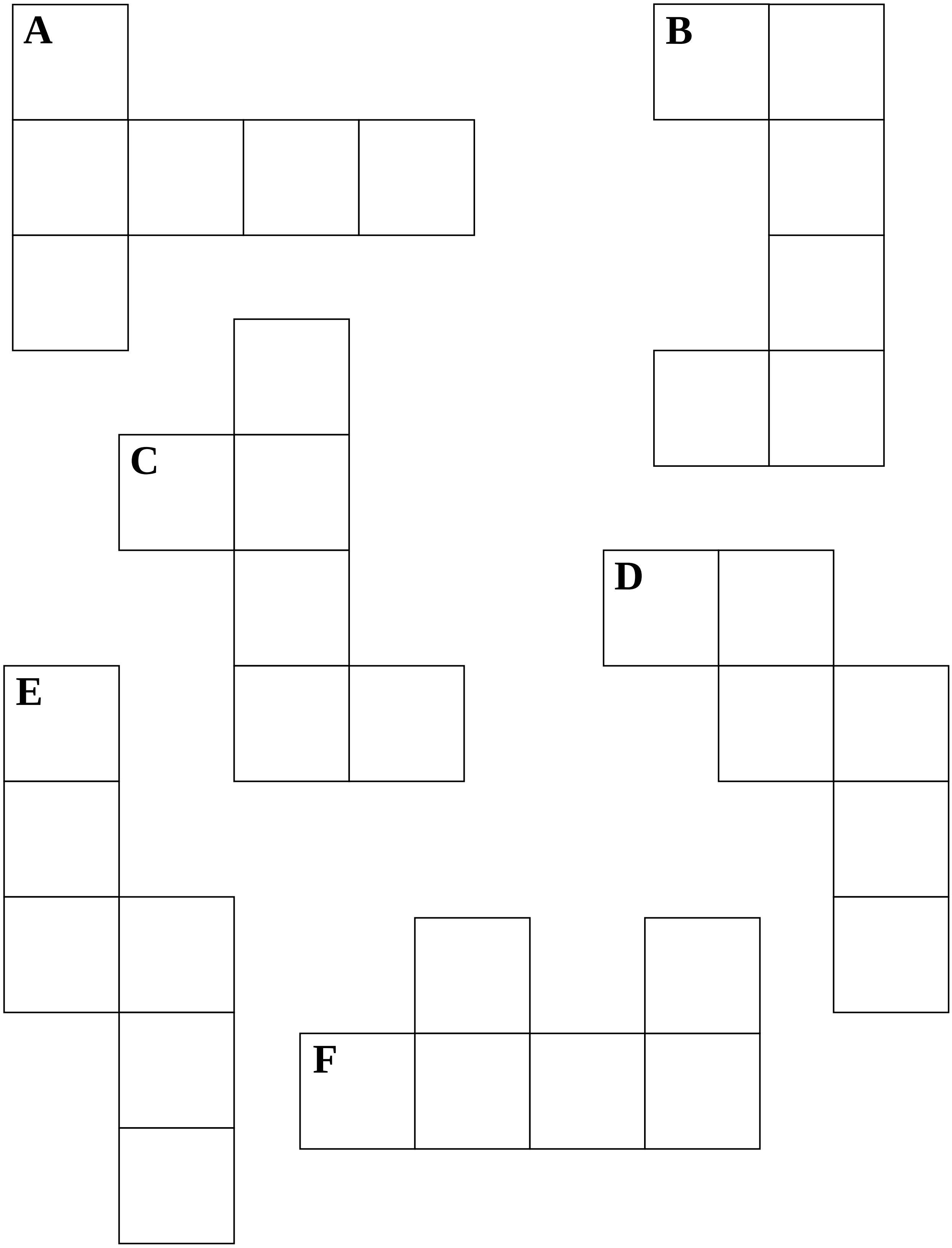




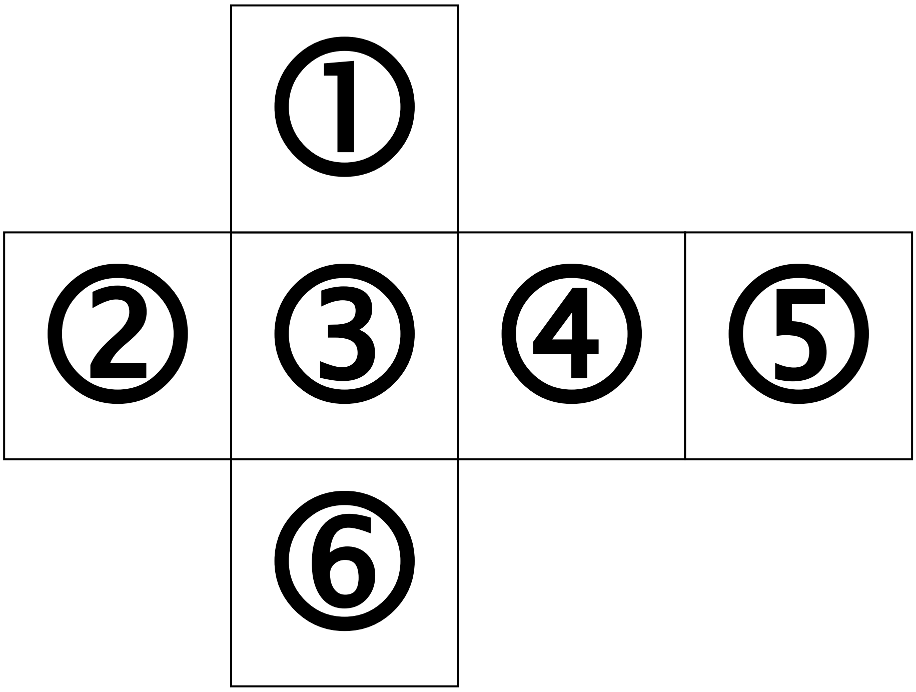


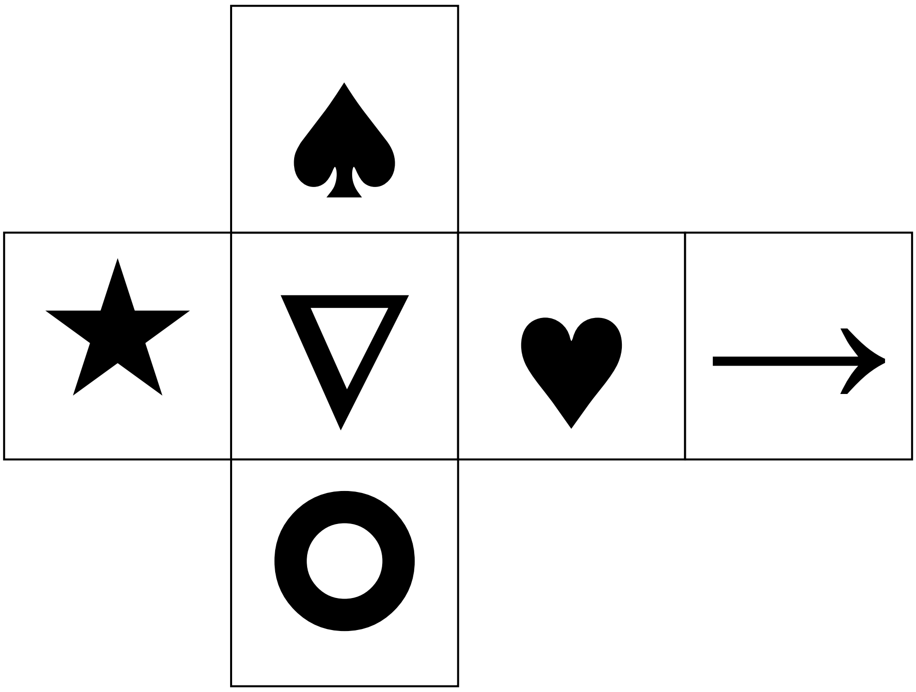
Cube or Not?

Cut out these shapes.
Will they fold up to make cubes?
Try to decide first, then cut them out and see.



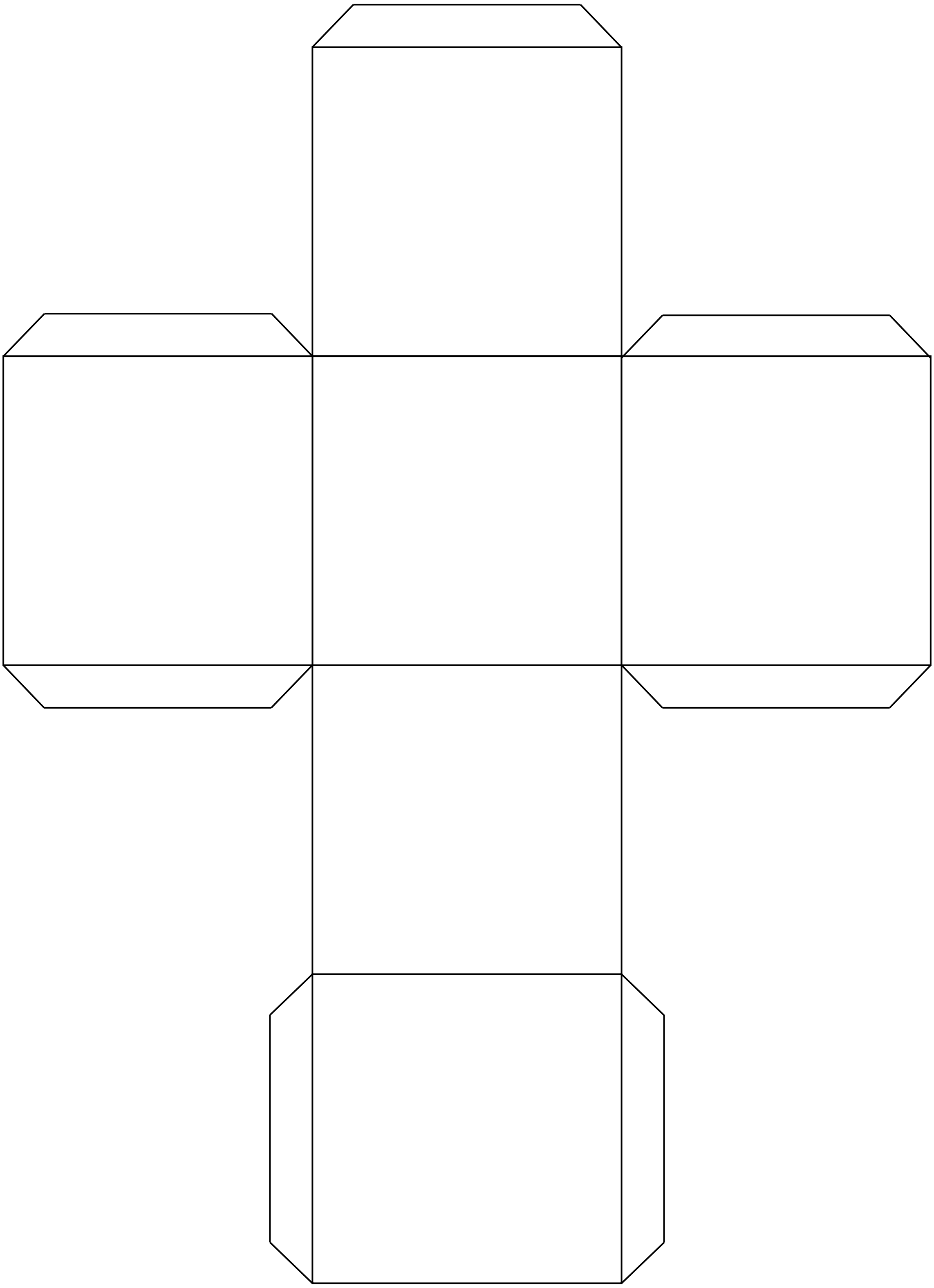
Why do some make cubes while others don’t?


①
②
③
④
⑤
⑥



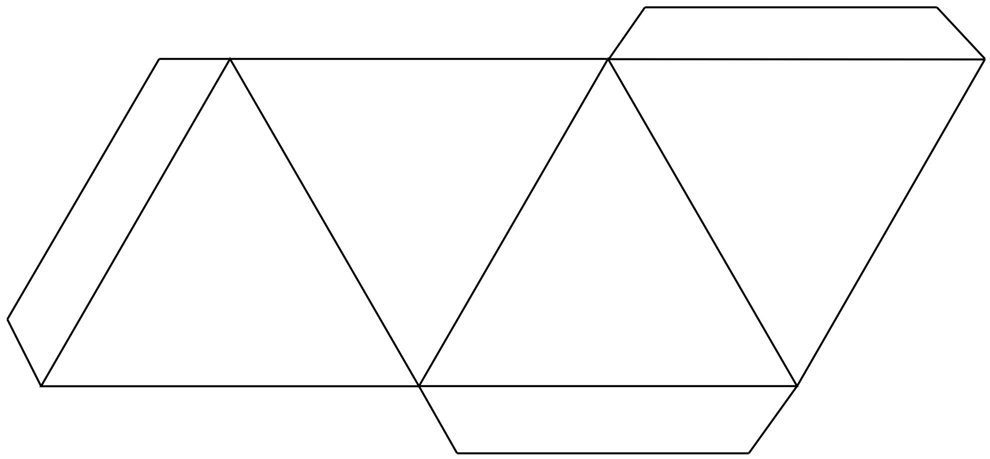
Cube

It's easier to put your name on before you fold it up!



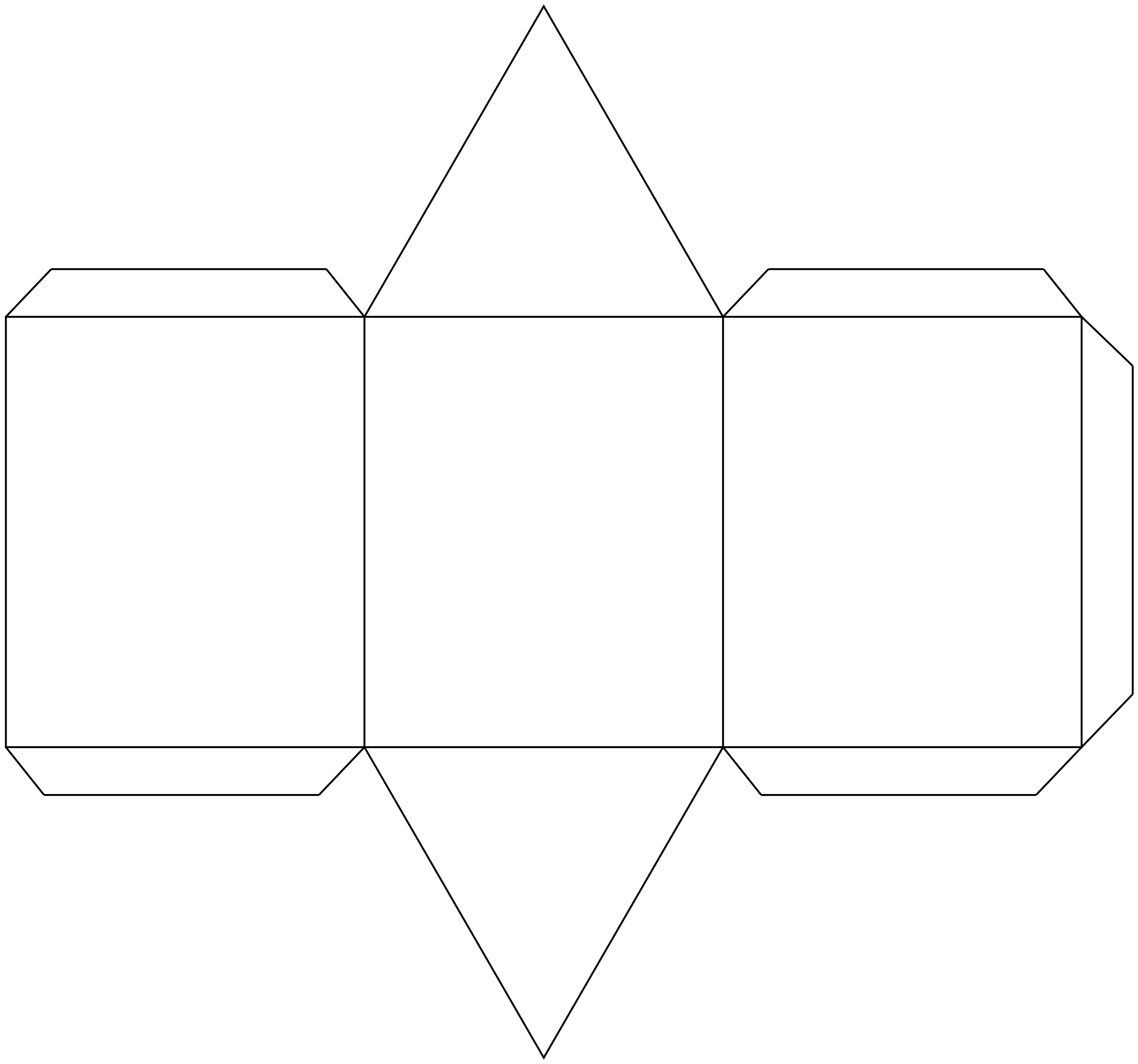
Tetrahedron

It's easier to put your name on before you fold it up!



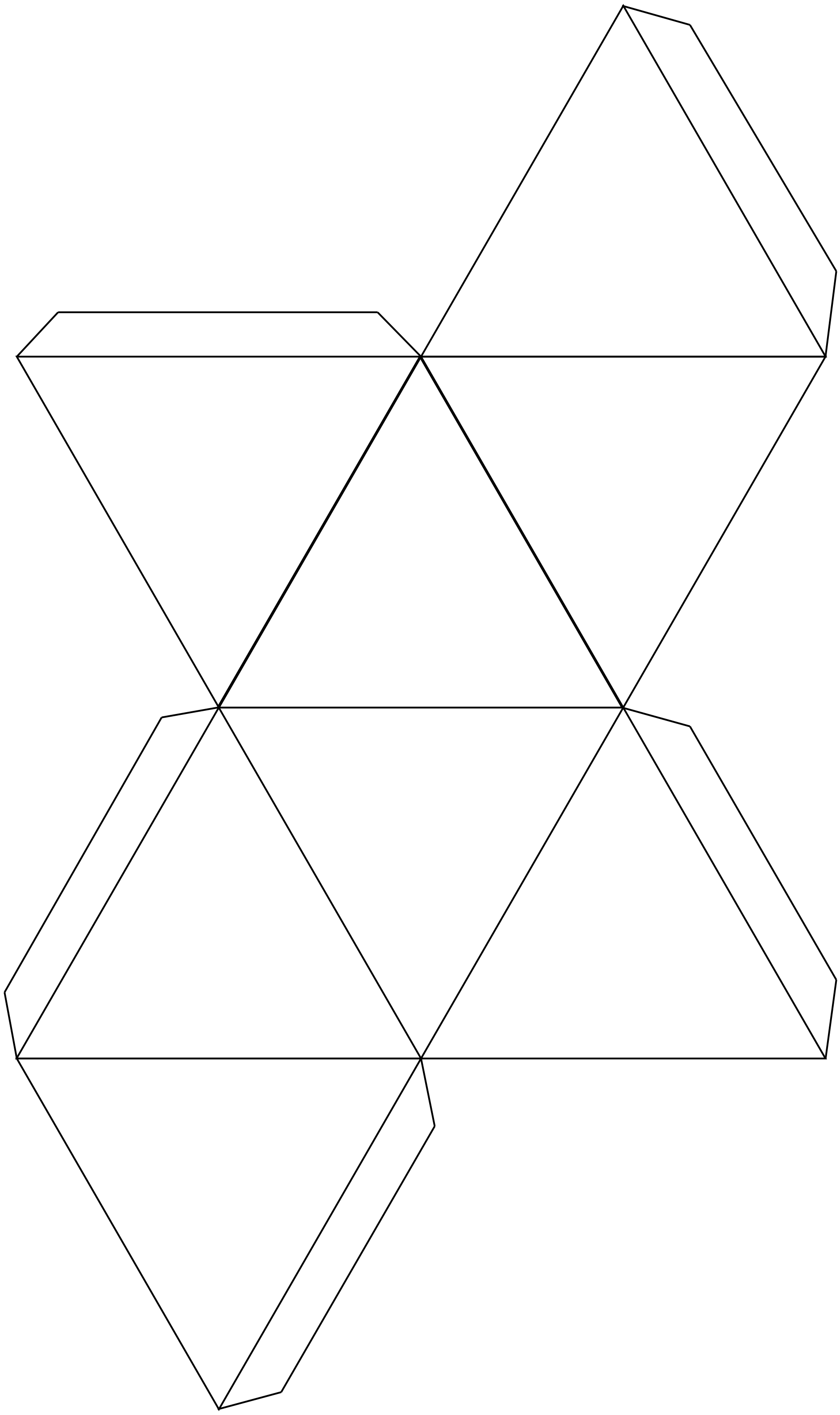
Triangular Prism

It's easier to put your name on before you fold it up!



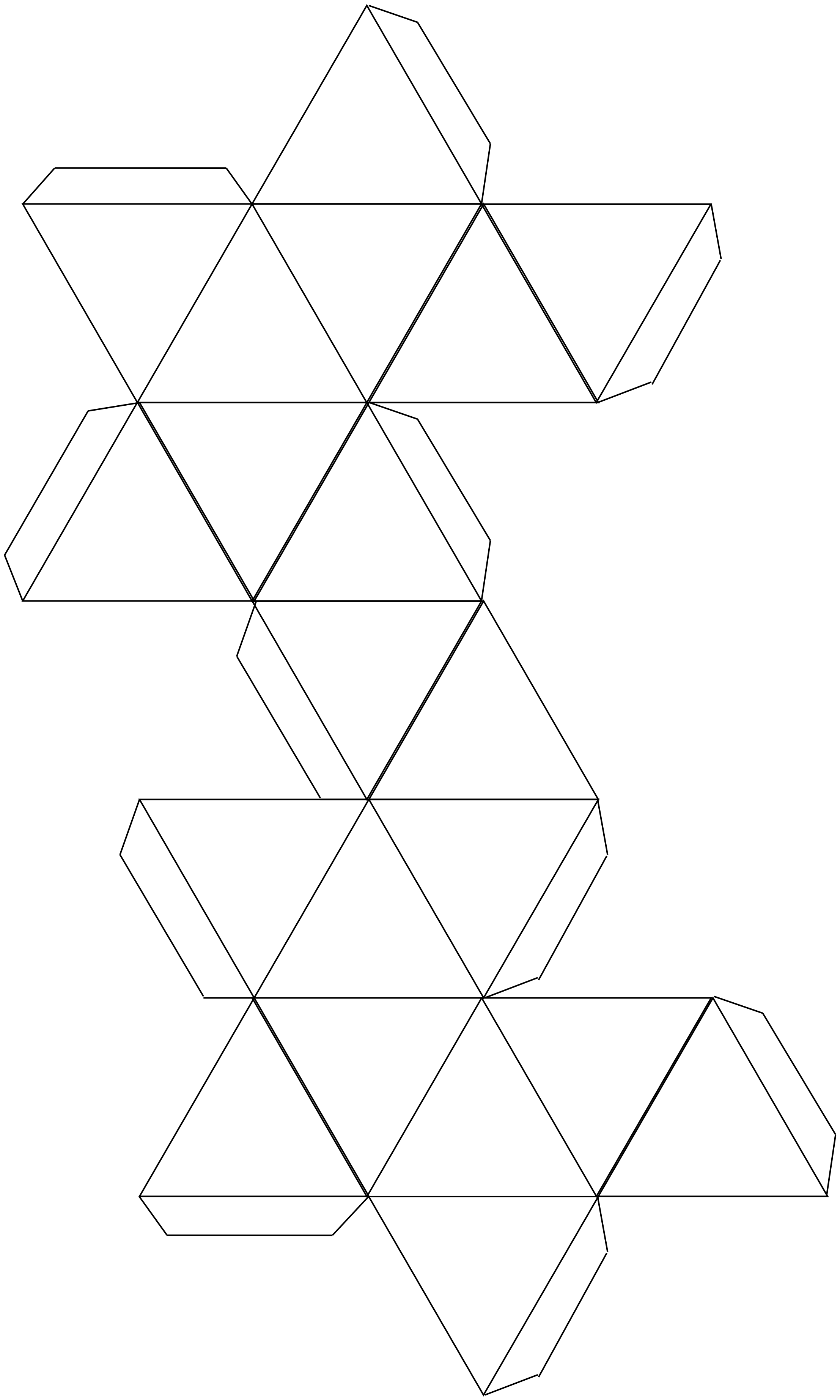
Octahedron

It's easier to put your name on before you fold it up!



Icosahedron

It’s easier to put your name on before you fold it up!



Dodecahedron

It's easier to put your name on before you fold it up!