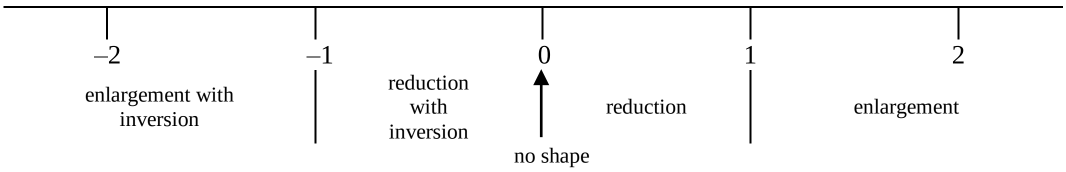
2.13 Symmetry

- Pupils need to be able to recognise and describe fully a transformation that's already happened.
  Certain information is needed to describe a transformation fully:

reflection	rotation	translation	enlargement
where is the mirror line? *give its equation if on co-ordinate axes*	where is the centre of rotation? what is the angle of rotation? what is the sense of the rotation? *positive (anti-clockwise) or negative (clockwise)*	how far and in what direction has the shape moved? *give the vector* $\begin{pmatrix}\textbf{distance to the right}\\ \textbf{distance up}\end{pmatrix}$	where is the centre of enlargement? what is the scale factor?

For enlargement you can draw a scale factor number line:



A scale factor of 1 leaves the shape unchanged; a scale factor of –1 inverts the shape but doesn't change its area.

- Pupils need to be able to perform a specified transformation of a shape. It's good to develop a culture where pupils check their own drawings by measuring lengths and angles so that they rarely need to ask "is this right?" Accuracy of ± 1 mm and ± 1° should be the aim.
- Marking is much easier when drawings are done on co-ordinate axes. Pupils can then write down the co-ordinates of the points of the image shape(s) and it is easy to see whether these are correct or not. Otherwise the only quick way to mark drawings is to do the accurate drawing yourself, photocopy onto an acetate (this is much easier – although more expensive - than trying to do an accurate drawing using acetate pens) and then place this over the pupil's work. You can set up some kind of rule for marks; e.g., within 1° or 1 mm, 2 marks; within 2° or 2 mm, 1 mark.
- Coloured acetate is available from art shops, and although it's expensive you can do a lot with just one sheet. You can cut out various shapes and place onto an acetate of 1 cm × 1 cm squares (available in section 1.23). This is particularly helpful with visualising translations. Dynamic Geometry software can make this even slicker.
- Small mirrors can be useful, and tracing paper is more or less essential for many pupils.

2.13.1	Reflections. Completing the other half of pictures of animals/aliens, etc. can be useful practice.	*Good for displays.*
	The reflected image is always *congruent* to the original object.	*Diagonal mirror lines are sometimes easier to do by rotating the paper so that the line goes away from you. You need to count the squares (or diagonals of squares) or measure in a direction perpendicular to the mirror line.* *"Diagonal" mirror lines that are not at 45° are very difficult to do accurately unless the shape is well chosen.*
	What stays the same and what changes in a reflection?	*Same: size, shape, lengths of sides, area, angles; Different: position, "orientation", "handedness".*
2.13.2	**NEED "**Symmetrical Squares" sheets.	*Several possible answers.*
2.13.3	Introduce by drawing axes from –6 to 6 in both directions on the board.	*Recaps plotting co-ordinates.*
	Plot the co-ordinates A(1,1), B(1,4), C(2,4), D(2,2), E(3,2) and F(3,1) and join them up to get an L-shape.	*Or you can use a scalene right-angled triangle. You don't want to use anything with symmetry because*

I'm going to add 3 to all the coordinates to get six new points. So A becomes (4,4). What do you think the new shape will be like?
We're adding 3 to the x-number (the first number) and 3 to the y-number (the second number).

that sometimes makes it hard to see if the shape has been changed or not, although so long as the vertices are labelled clearly this does not have to be a problem.

Many will think shape will be stretched or enlarged.

Actually it's just a translation $\begin{pmatrix}3\\3\end{pmatrix}$.

So transforming the co-ordinates (x, y) *into* $(x+3, y+3)$ *is the translation* $\begin{pmatrix}3\\3\end{pmatrix}$.

What if leave the x-numbers alone and make the y-numbers into *minus* what they are?
i.e., $(x, y) \rightarrow (x, -y)$

This time it's a reflection in the x*-axis.*

Put up a list of possible co-ordinate transformations.
Pupils can invent their own.
They could work in groups so as to cover all these as a class in a reasonable amount of time.
Make a table of results.

$(x, y) \rightarrow$	***transformation***
$(x+3, y+3)$	*translation 3 to the right, 3 up*
$(x, -y)$	*reflection in* $y = 0$
$(-x, y)$	*reflection in* $x = 0$
$(-x, -y)$	*rotation 180° about (0,0)*
(y, x)	*reflection in* $y = x$
$(y, -x)$	*rotation –90° about (0,0)*
$(-y, x)$	*rotation +90° about (0,0)*
$(-y, -x)$	*reflection in* $y = -x$
$(x+1, y-3)$	*translation* $\begin{pmatrix}1\\-3\end{pmatrix}$
$(2x, 2y)$	*enlargement, scale factor 2, centre (0,0)* ***[Need vertical axis up to 8 for this one.]***

Try to generalise; e.g.,
$(x+a, y+b)$ is a translation $\begin{pmatrix}a\\b\end{pmatrix}$
(even if a or b are negative).

(ax, ay) is an enlargement, scale factor a centred on the origin.

Try out more complicated ones; e.g.,
$(x, y) \rightarrow (3x-2, 3y+1)$.
An enlargement, scale factor 3 about the origin followed by a translation 2 units to the left and 1 unit up.

2.13.4 An alternative approach is to use Dynamic Geometry software to allow pupils to explore different transformations on a shape of their choice and investigate what happens to the co-ordinates of the vertices under each different transformation.

The software may allow other transformations such as stretch and shear.

Some pupils might like to investigate matrices to try to work out the effects of putting different numbers in the four different "slots".

2.13.5 Is a human face symmetrical?

There is more or less a vertical line of symmetry, but not exactly. If it were, our faces would look the same in the mirror, and they don't. Some studies suggest that highly symmetrical faces are the most beautiful.

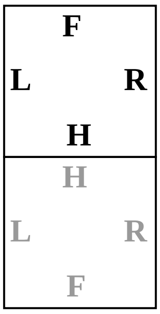
What's the minimum change you'd have to make to a human face to give it some rotational symmetry?! (Pupils can sketch their ideas.)

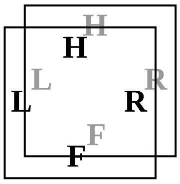
You could aim for order 2 rotational symmetry, and even that is not easy.

2.13.6 Why does a mirror swap round left and right but it doesn't swap round up and down?
I mean why is my left hand where my right hand is (and vice versa), but my head isn't where my feet are (and vice versa)?

Answer: This is quite a tricky one.
It's really because we imagine our mirror image standing (upright) next to us.
If you're facing a mirror straight-on, every point is reflected exactly in front of the original point.

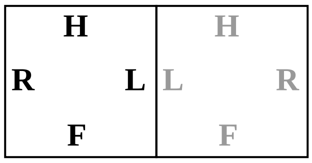
Does a mirror really know which way is up? (What we mean by "up" is really something like "the opposite way to gravity" – how could a mirror know about gravity?)





(Black for the person; grey for the reflection; H= head, F = feet, L = left hand, R = right hand.)

But to compare the original with the image, you have to turn one of them over so you can place them side by side. We tend to rotate ourselves mentally about a vertical *axis, and that gives the diagram below,*



where R and L have swapped, but that's really an arbitrary choice.
If we rotated about a horizontal *axis, we would get the opposite result (see left) where the head and feet have swapped places.*

2.13.7 Rotations
Order of rotational symmetry is the number of times the shape will fit onto itself as it rotates through 360°.
Order 1 means no rotational symmetry.
A circle has infinite rotational symmetry, because you can stop it at any angle and it fits exactly onto itself.

To explain order of rotation you can use interlocking cubes which have holes in the middle. A pencil will fit through horizontally so that you can rotate the object about that axis.

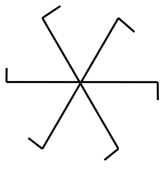
Order is always a positive integer. (Actually, in quantum mechanics some particles – for example, an electron – have so-called "spin" of $\frac{1}{2}$, and this means they have to "rotate twice", 720˚, to get back to where they started!)

To rotate a shape without using tracing paper it often helps to join the centre of rotation to one of the vertices and rotate this line. You can do this for each vertex if necessary.

If using tracing paper, "centre of rotation" can be seen as the point on the tracing paper where you put your pencil – the point that doesn't move – everything else revolves around it.

Draw me a shape with order 6 rotational symmetry, but no reflection symmetry.

Lots of possibilities – you can lose the mirror symmetry by adding tails or flags to something with 6-fold symmetry.
e.g.,



You may want to avoid unintentionally drawing Swastikas.

What stays the same and what changes in a rotation?

Same: size, shape, lengths of sides, angles, area;
Different: orientation, position.

2.13.8 **NEED** a set of circuit symbols from a Physics/Electronics book.
What kind of symmetry do the symbols have?

Line symmetry may be parallel or perpendicular to the wire direction.

Answers: among the most common symbols, you find these symmetries:

- *no symmetry: variable resistor, variable capacitor, transistor, switch;*
- *line symmetry only: cell, ammeter, voltmeter, earth, diode, inductor, lamp (modern symbol);*
- *rotational symmetry (> order 1) only: source of alternating current;*

- *line symmetry and rotational symmetry (> order 1): connecting wire, lamp (old-fashioned cross symbol), resistor, transformer, fuse, capacitor.*

You can do the same with hazard warning symbols (Science department) (ignore the writing underneath the symbol).

"Toxic" has line symmetry, and "harmful" and "radioactive" have both line and rotational. "Oxidising" almost *has line symmetry but not quite because of the "flames"!*

Highway code road signs are another possibility, but most have no symmetry. Generally you should ignore the writing underneath and possibly ignore the shape of the sign itself (triangle, circle, etc.) as well.

- *line symmetry only: crossroads, dual carriageway ends, chevrons, road narrows on both sides, uneven road, traffic signals, hump bridge, level crossing with barrier, general danger, tunnel, low-flying air-craft, road humps;*
- *line symmetry and rotational symmetry: general warning, roundabout (line symmetry only approximate here).*

Signs from music notation.

- *line symmetry only: accents, ties, pause, crescendo, diminuendo, up/down bow (string players), alto/tenor clef;*
- *rotational symmetry only: sharp sign, natural sign, turn, mordent;*
- *line symmetry and rotational symmetry: breve, semibreve (and their rests), 5-line stave, bar line, double bar line, repeat marks, staccato dots, double-sharp sign.*

2.13.9 **NEED** crosswords from newspapers (collect for homework). Sort them according to their symmetry.

You could make a display out of this.

2.13.10 **NEED** pencil crayons (or just pencil), sheets.
Colour And Symmetry.
Colour in the shapes to give them rotational symmetry of

1. order 1
2. order 2
3. order 3

Answers:
Colouring the hexagon in the centre never makes any difference so long as it's all the same colour.

See sheet for answers.

What other orders of rotational symmetry are possible?

None.

What is the minimum number of different colours you need to use?

One (and white).

Do any of the finished shapes have any line symmetry?

No.

If using different colours, be careful not to embarrass anyone who is colour-blind.

2.13.11 Translations.
Pupils need to be clear that the vector defines the movement of each point to its image point; this isn't necessarily the same as the "gap" between the object and image shapes.

Translation vectors are not that difficult and are a less cumbersome way of describing translations than using words.
They are best defined as

$$\begin{pmatrix}\textbf{distance to the right}\\ \textbf{distance up}\end{pmatrix}.$$

(Notice that this is "upside down" compared with the way gradient is defined.)

What stays the same and what changes in a translation?

Same: size, shape, lengths of sides, area, angles, orientation;
Different: position.

2.13.12 Combined Transformations.

1. I'm thinking of a point. If I translate it by $\begin{pmatrix}4\\0\end{pmatrix}$, I get to the same point as if I reflect the point I'm thinking of in the y-axis. Where is the point? (There is more than one possibility.)

2. I'm thinking of a point. If I translate the point by $\begin{pmatrix}3\\-3\end{pmatrix}$, I get to the same point as if I reflect the point I'm thinking of in the lines $y = x$. Where could the point be?

3. I'm thinking of a point. If I translate the point by $\begin{pmatrix}-2\\-6\end{pmatrix}$, that's equivalent to a rotation of it by 90° clockwise about the origin. Where could the point be this time?

4. This time if I translate my point by $\begin{pmatrix}4\\2\end{pmatrix}$, that's equivalent to rotating it by 90° clockwise about the origin. Where is this point?

Answers:

1. (–2, anything); i.e., any point on the line $x = -2$;
2. either (1,4) or (–4,–1);
3. (4,2)
4. (–3,1)

Solve these by doing rough sketches.

In general, for questions 3 and 4, if a translation $\begin{pmatrix}a\\b\end{pmatrix}$ is equivalent to a rotation 90° clockwise about the origin, then the co-ordinates of the point have to be $(-\frac{1}{2}(a+b), \frac{1}{2}(a-b))$.

2.13.13 Enlargement.
What does "enlargement" mean?
Draw a 3 × 2 rectangle on the board.

"Gets bigger" – so draw a 10 × 2 rectangle;
"Gets bigger both ways" – so draw a 10 × 10;
"Gets bigger both ways by the same amount" – so draw a 4 × 3 rectangle, etc. (be awkward!).

Why are none of these proper enlargements?

It has to get the same proportion *(fraction) bigger both ways.*
Proportional thinking is always hard.

If the "scale factor" is different in different directions, you get a stretch. You wouldn't be happy with this if your photos got "enlarged" like this – it isn't a proper enlargement.

What stays the same and what changes in an enlargement?

Same: shape, angles, orientation;
Different: size, position, lengths of sides, area.

A reduction *sometimes counts as a (fractional) enlargement in maths.*

The scale factor number-line may be helpful here (see beginning of this section).

2.13.14 Accurate Enlargements.
You don't always need to have a centre of enlargement to draw an accurate enlargement; e.g., you can measure the sides and angles, keep the angles the same and multiply the lengths of sides by the scale factor.

In fact there isn't always a centre of enlargement even when a shape has been enlarged properly, because the new shape could have a different orientation from the original shape.

Scale factor (SF) can be positive or negative.

Initially it's useful to use photocopied sheets so that you can be sure the enlarged shape will fit on nicely (see sheet).

Emphasise that we make every measurement from the centre of enlargement. (If you measure from the corners of the original shape instead you get a SF + 1 enlargement.)

What difference does it make if we move the centre of enlargement?
Same image shape except in a different place.
(Centre of enlargement can even be inside the shape or on one of the vertices.)

Pupils should check their own drawings by measuring the sides in the new shape (they should be SF × the lengths of the corresponding sides in the old shape), the angles (should be the same) and *checking that the orientation is the same.*

Four possible "kinds" of scale factor (SF):

1. SF >1; shape gets bigger;
2. 0 < SF < 1; shape gets smaller;
3. SF < –1; shape gets bigger and inverted;
4. –1 < SF < 0; shape gets smaller and inverted.

See the SF number-line at the beginning of this section.
Probably best to do in this order.

2.13.15 Enlargement. "Aspect Ratios", TV/cinema.
A normal TV screen has an "aspect ratio" of 4:3 (its size is 4 along by 3 up).
If you display a widescreen movie (2.35:1) so that the whole screen is filled with picture, what % of the picture do you lose?

Answers:
% viewed = $\frac{4\times1}{2.35\times3} = 57\%$ *, so 43% is missing.*
(You see all of the vertical direction but lose the two ends in the horizontal direction.)

What about if you view the whole picture (so you don't miss anything) "letterbox" style. What % of the screen is wasted with "black bars"?

Same as before; 43% of the screen is black.

(Widescreen isn't always as "wide": 1.85:1 is common, as is 16:9, which is normal theatre screen dimensions.)

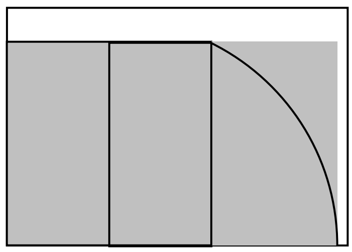
Which do you think is better?

Film buffs tend to prefer to see everything the director intended, even if that means having a smaller picture.

What if you have a high-definition TV (16:9)?

This time you lose only 24% of the picture (or waste 24% of the screen).

2.13.16 **NEED c**ompasses, A4 plain paper.
Constructing a Golden Rectangle.
Take piece of A4 paper, landscape orientation, and draw a square 18 cm by 18 cm in the bottom left corner.
Split the square into two congruent rectangles with a vertical line.
Place the point of your compasses at the bottom of this line and stretch the pencil up to the top right corner of the square.
Draw an arc down from here until it reaches the bottom of the paper.
This point along the bottom side is the position of the bottom right end of the Golden Rectangle.
From here, draw a line 18 cm long vertically up the page. Then draw a line to meet the left side of the paper.



The shaded rectangle is called the "Golden Rectangle". Its sides are in the ratio $1:\phi$ *where* ϕ *= 1.61803... (*$\frac{1+\sqrt{5}}{2}$ *, see below).*
(Pythagoras' Theorem gives the radius of the arc as $9\sqrt{5}$ *, so the bottom length is* $9(1+\sqrt{5})$ *.)*

2.13.17 **NEED** A4 white paper. Golden Ratio.
Suitable homework. Draw 8 different-shaped rectangles on a blank piece of A4 paper (or cut out 8 different rectangles). Ask people to say which one or two look the "nicest" – "most pleasing to the eye".

Answer: People tend to choose the ones nearest to 3:2 or thereabouts. Some say that the rectangle most pleasing to the eye is the Golden Rectangle with sides in the Golden Ratio ($1:\phi$*, see section 2.13.16).*
Renaissance artists may have used this to construct paintings.

2.13.18 **NEED** scrap paper, scissors.
I take a rectangle and fold over the shorter end so that it lies along the longer side. In this way I can mark off a square from the end. I cut off the square and the rectangle I'm left with, although it's obviously smaller, is the same shape (same dimensions) as the rectangle I began with. Can you find a rectangle that will do that. Will a 2:1 rectangle work?

Answer:
2:1 will give 2 squares, so certainly not. If the sides are in the ratio $x:1$ *(* $x>1$ *), then algebraically*
$\frac{x-1}{1} = \frac{1}{x}$ *, so* $x(x-1)-1=0$ *or* $x^2-x-1=0$ *, and the solutions are*

$$x = \frac{1\pm\sqrt{1--4}}{2}, \text{ or } \frac{1+\sqrt{5}}{2},$$

since x *must be positive.*
So $x = 1.61803...$
This is the "divine proportion" or "Golden Ratio".
As indicated above, it is squared by adding 1

(See task involving A-size paper in section 1.10.6.)

($x^2 = x + 1$).

(See section 1.19.10 for a task involving the Fibonacci series.)

The ratio of a term in the Fibonacci sequence (1170-1250) to the previous term gets closer to the Golden Ratio as you go to higher and higher terms.

2.13.18 **NEED** A3 piece of paper showing a large footprint or "pawprint". What can you say about the size of the animal that could have produced this?! (Imagine we discovered it outside school in the morning.)

Pupils can take measurements from it and try to predict things like height, mass, length of stride, the tallest wall it could climb over, how much food it might eat per day, etc.

2.13.19 On squared board or 1 cm × 1 cm squared acetate, draw two separate 2 × 2 squares.

Suitable for reviewing the transformations topic.

Lots of possibilities: translation, rotation, reflection followed by translation, reflection followed by a different reflection, etc.

Pupils can try to describe the transformations as precisely as possible.

The white square has become the grey square. What's happened to it, apart from the change in colour?

Label the white square ABCD. How would you have to label the grey square (where would you put A', B', C' and D' to make it each of the transformations pupils have suggested?

The white square is the object; the grey square is the image.

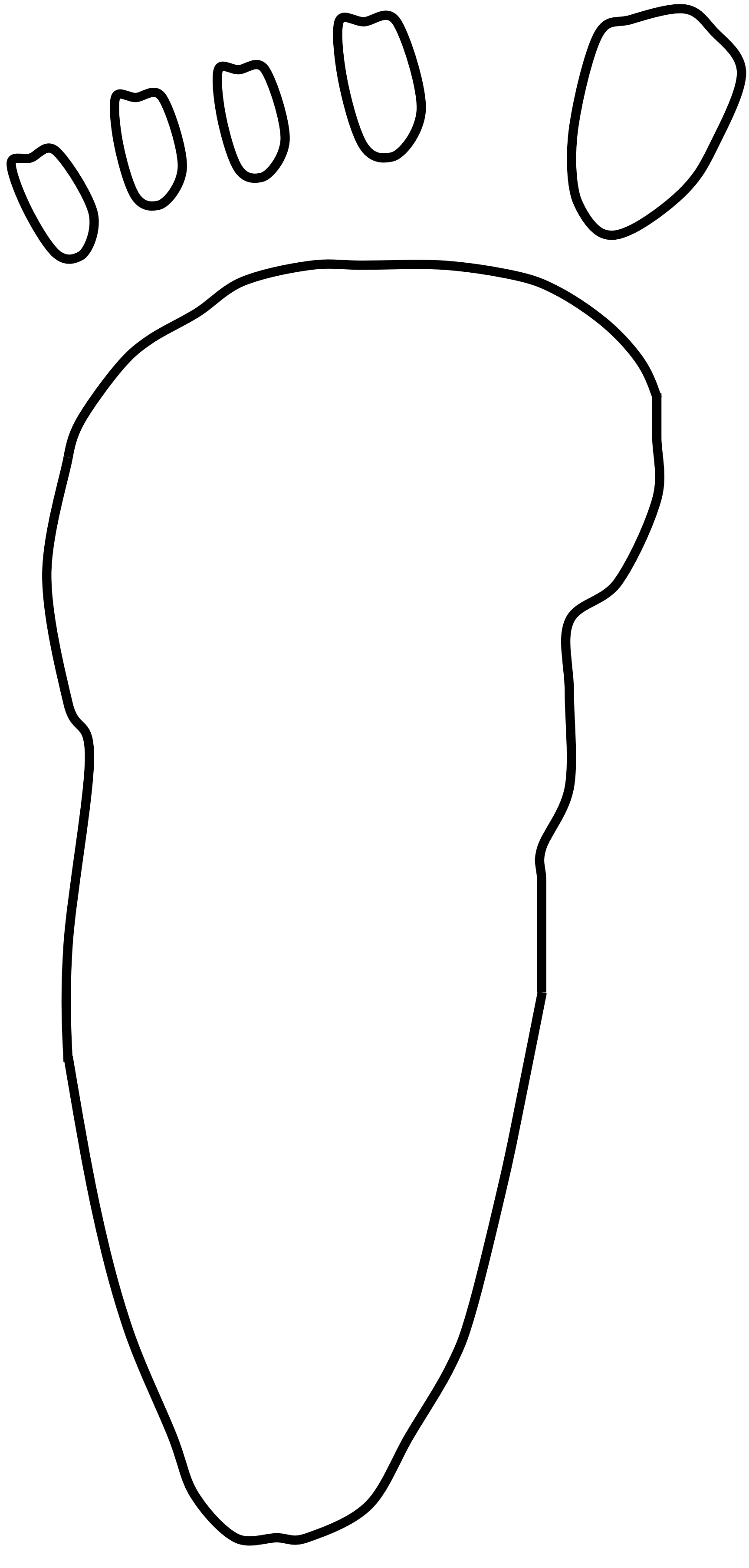
2.13.20 You could begin a lesson by writing something like this on the board:

What do you think this lesson is going to be about?

Be prepared to help those who may find this very hard.

You can give instructions in this way to pupils at the start of a lesson on reflections; e.g.,

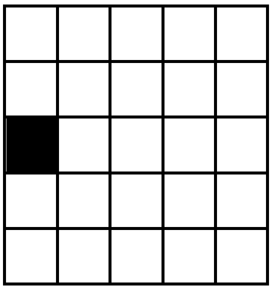
Don't say anything.
If you can understand this, then turn to the back of your exercise book and write your first name and last name in reflection writing.



Symmetrical Squares

In each drawing, shade in *exactly 3 more* squares so that the whole drawing ends up with *exactly 2 lines* of symmetry.

1



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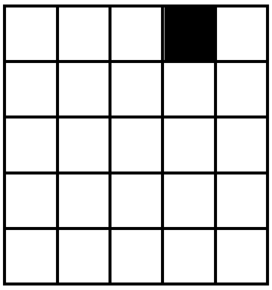

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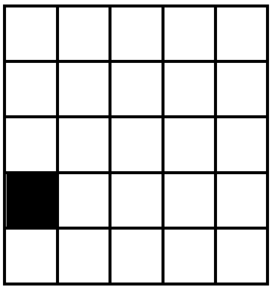

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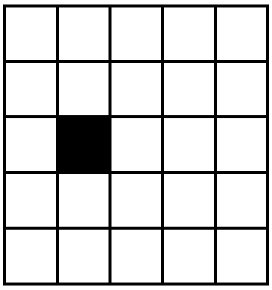

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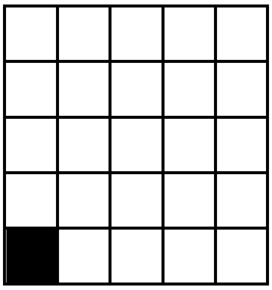

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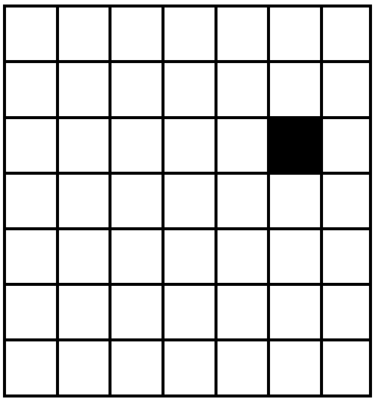

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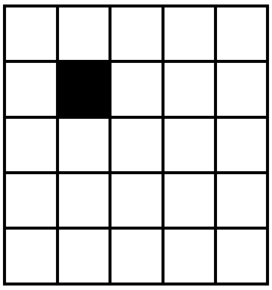

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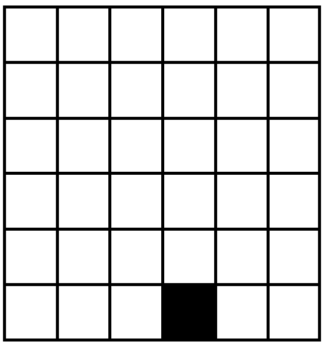
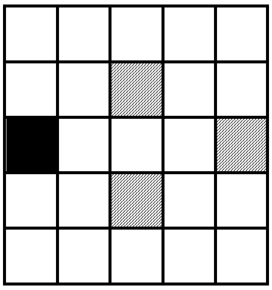


Symmetrical Squares ANSWERS

In each drawing, shade in *exactly 3 more* squares so that the whole drawing ends up with *exactly 2 lines* of symmetry.

There are many possibilities; only one answer is shown for each question.
All of the shaded squares would have to be the same colour.

1



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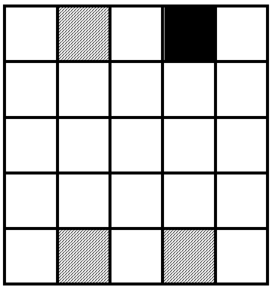

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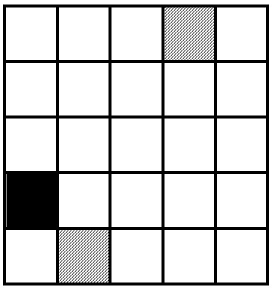

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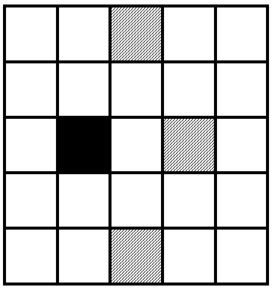

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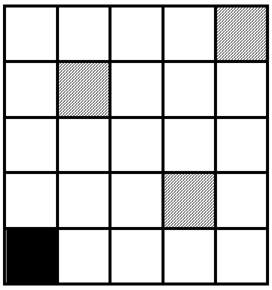

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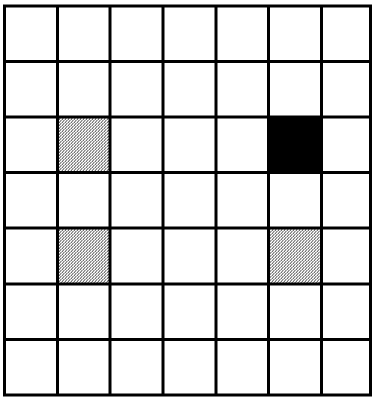

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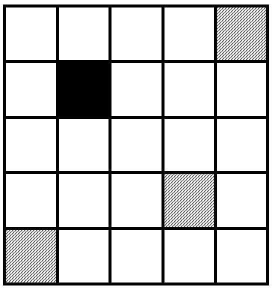

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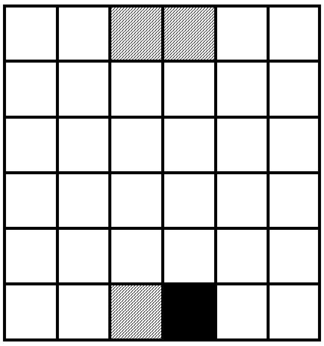
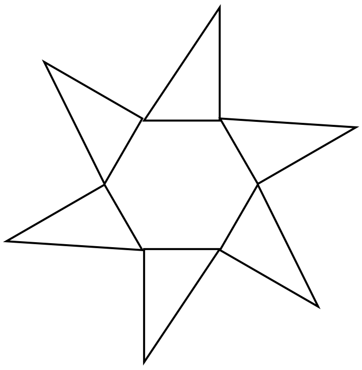
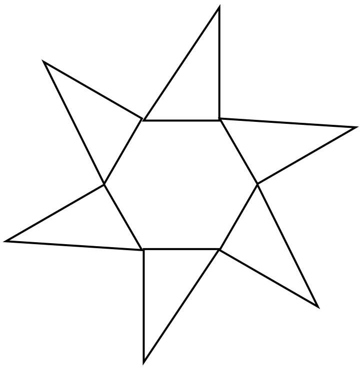


Colour And Symmetry

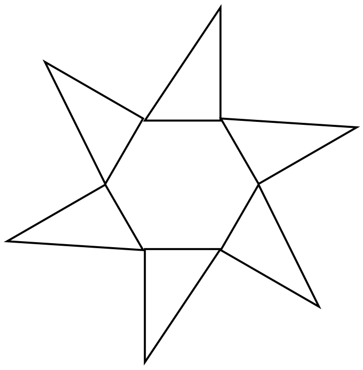
Colour these shapes so that they have different orders of rotational symmetry.



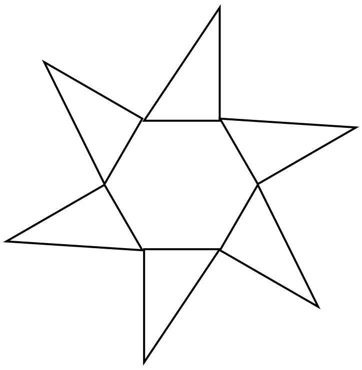
Order ______



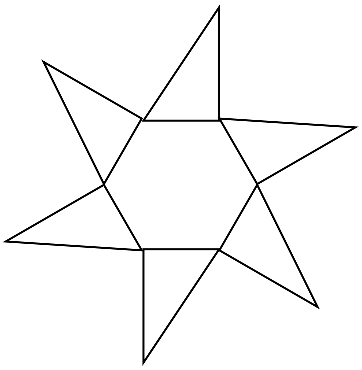
Order ______



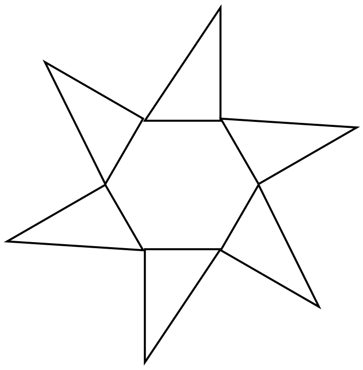
Order ______



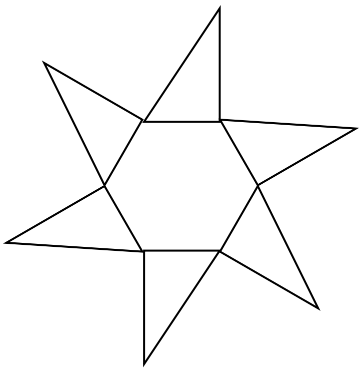
Order ______



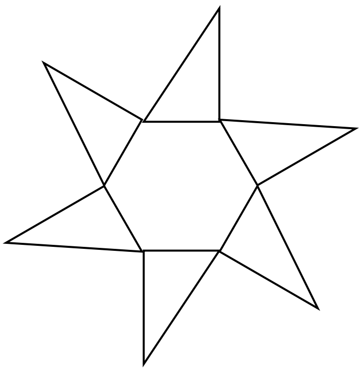
Order ______



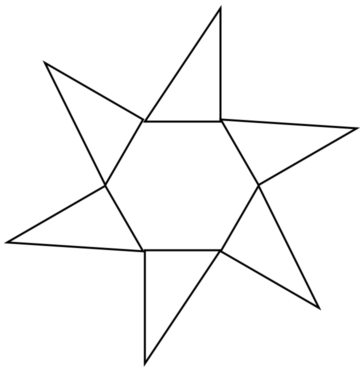
Order ______



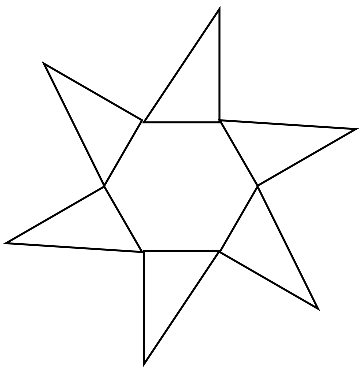
Order ______



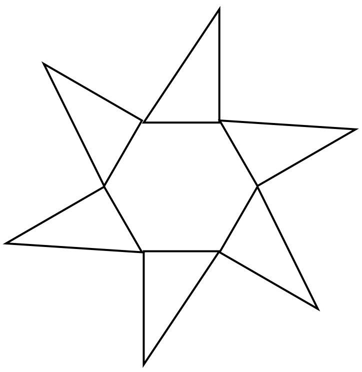
Order ______



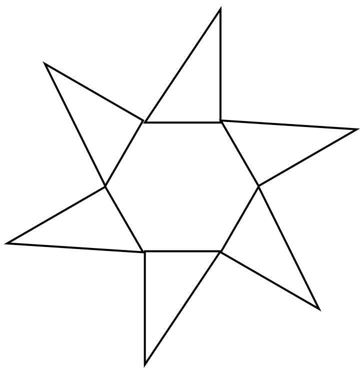
Order ______



Order ______



Order ______

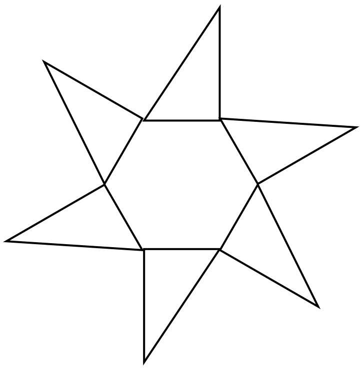


Order ______

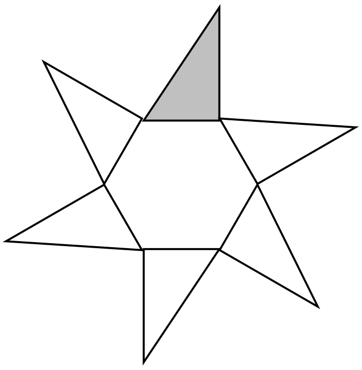
Colour And Symmetry ANSWERS

Colour these shapes so that they have different orders of rotational symmetry.

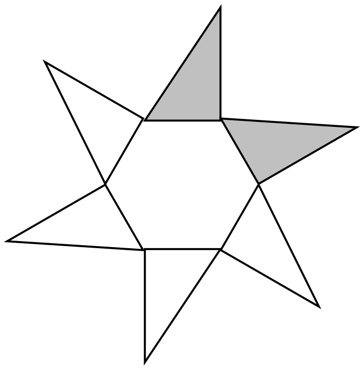
These are the possibilities using just one colour (and white).
In each case, the shaded and white areas could be swapped (making a "negative"), usually giving a different answer with the same order of rotational symmetry.



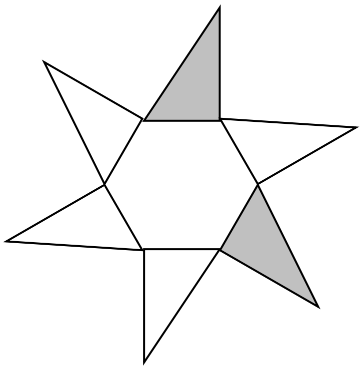
Order 6



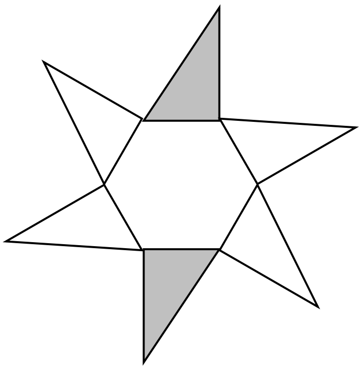
Order 1



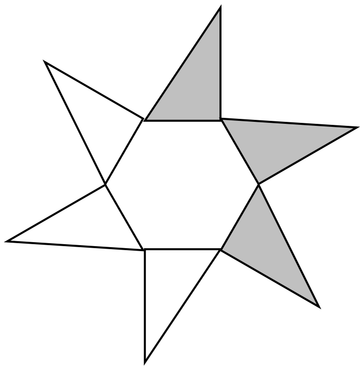
Order 1 ("ortho")



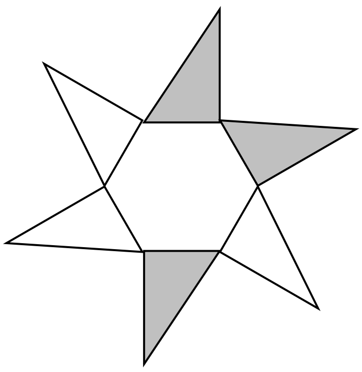
Order 1 ("meta")



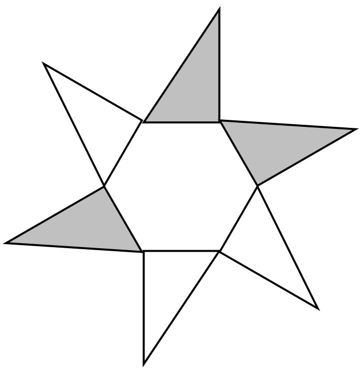
Order 2 ("para")



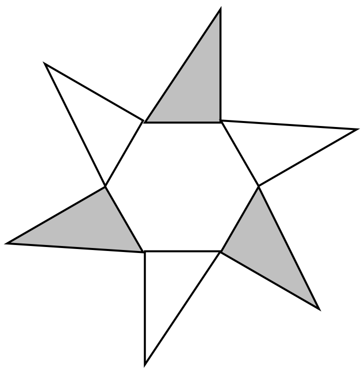
Order 1



Order 1



Order 1



Order 3

These two (above) have "negatives" which are the same as themselves but they are not the same as each other.

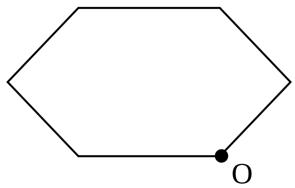
The names "ortho", "meta" and "para" refer to substitution patterns in derivatives of the chemical molecule benzene, which has a planar hexagonal shape and 6-fold symmetry.

Drawing Accurate Enlargements

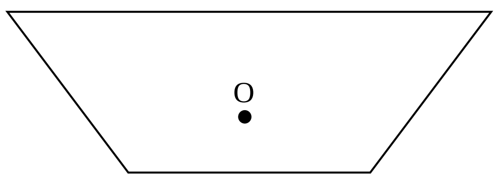
Enlarge these shapes as accurately as you can, using O as the centre of enlargement.
None of the enlargements should go off the page.

1 Scale factor 2

2 Scale factor 3



3 Scale factor 2



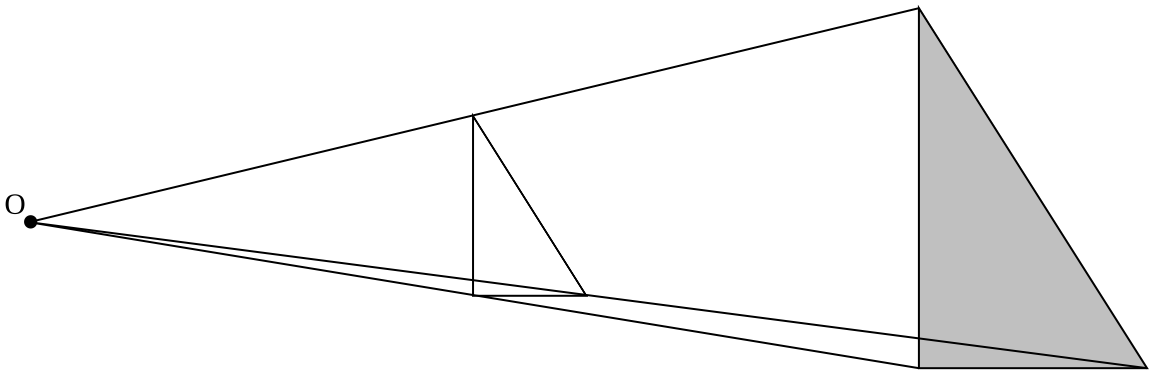
4 Scale factor 1.5

Drawing Accurate Enlargements **ANSWERS**

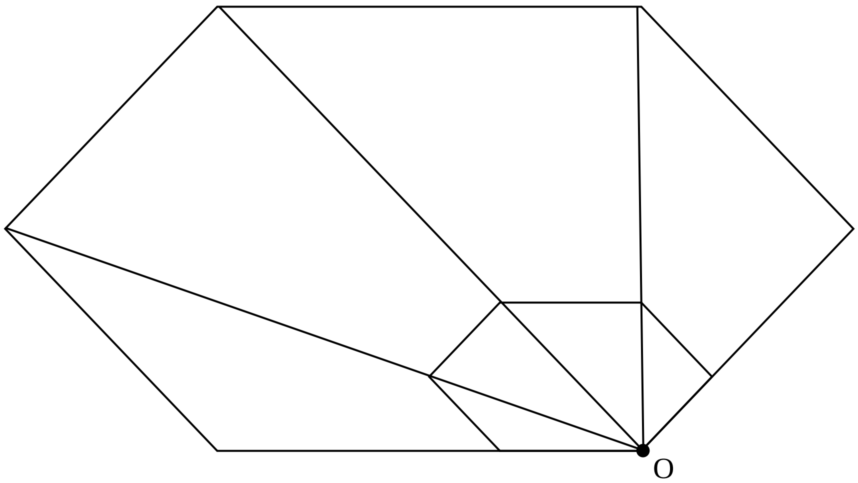
Enlarge these shapes as accurately as you can, using O as the centre of enlargement.
None of the enlargements should go off the page.

(You could photocopy this sheet onto an acetate and place it over the pupils' work to mark.)

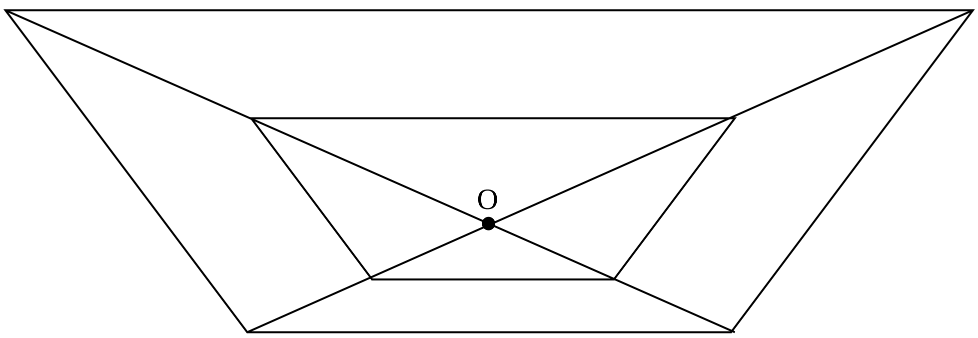
1 Scale factor 2



2 Scale factor 3



3 Scale factor 2



4 Scale factor 1.5