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FREEDOM AND CONSTRAINT

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Every year for the last five years a small conference has been held at which some 20 people discuss a theme in mathematics education over 4 days. We call this the Institute of Mathematics Pedagogy. It is a little like a very long, intense, ATM conference session: an intellectual, emotional and social workshop during which we do maths, talk, eat, play, dream and write, with the sole aim of understanding more.

In 2004 the theme was 'freedom and constraint'. We found that this theme permeated throughout our work, whatever our role as educators might be. This article is a compendium of our thoughts.

In mathematics the constraints of axioms, laws and properties are required to solve problems and to understand relationships. These constraints are so implicit in the maths we learn that most of the time they seem intuitive. We know about squares, we know about rectangles, and hence we can think about what happens if we manipulate them, or compare them, or look at the algebra of their diagonals, or the relationships between their parameters. These general constraints provide a backbone for mathematics learning.

As we begin to develop mathematical knowledge, however, we can take control of the constraints. A square will always be a square, so long as it has certain properties that cannot change. But I can play with it and vary its size, position and orientation. What happens if I reflect it, rotate it, enlarge it, place it in a circle, give it area *x*? What happens if I try to construct one using only one side and perpendicular and parallel lines as construction tools? What can vary and it still be a square? What do I have to vary to make it NOT be a square?

Mathematics is enlivened both by specific constraints in a problem context and by specific variations in a playful context. Of course, for square you could read any other mathematical object, and apply the same thinking.

Within a particular mathematical context, imposing conditions will typically lead to new freedoms and constraints: if you are drawing 2-dimensional shapes and I restrict you to polygons, you are drawn into a world of possibilities you might not have considered before. If you were already thinking of polygons, then imposing 'polygon' as an explicit constraint may open up other possibilities (what if not . . .?). If you were already thinking about wild and wiggly shapes in 2D, shapes with infinite numbers of sides and vertices, shapes which do not close, and so on then the 'polygon' condition provides you with a new kind of freedom to explore within the obvious constraint.

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Of course, mathematical structure contributes its own constraints: if you are drawing triangles on a flat plane you are constrained to have the internal angles add up to 180° whether you like it or not. Other conditions are more arbitrary; for instance, we could require one of the angles to be 90°. Generally these conditions are imposed either by somebody else (say, a teacher as a means of directing work into a particular area, such as trigonometry) or by the bounds of mathematical possibility. When we impose conditions on students' mathematics, are we explicit about the different nature of these constraints? How do constraints on definitions and constraints on properties feel while you play? Are they different?

In what sense can we choose constraints? In

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Are students clear about who is exercising the power: is it the teacher, or the student or the mathematics?

what sense do we choose constraints? How is freedom related to awareness of constraint? How can students become aware that base 10 is a constraint, rather than 'just the way it is', and that they are free to explore other number bases? Are students clear about who is exercising the power: is it the teacher, or the student or the mathematics?

One task we worked on which illustrates these aspects of freedom and constraints, is the following, which came originally from John Mack in Sydney.

Imagine a circle. Let it move around on your mental plane so that you are aware of its freedom: you can vary the size and the position. The positions which the centre of the circle can take up provide one way to think about the extent of the freedom of the circle: the centre can be anywhere on the plane, and there are lots of choices of circle once the position of the centre has been chosen.

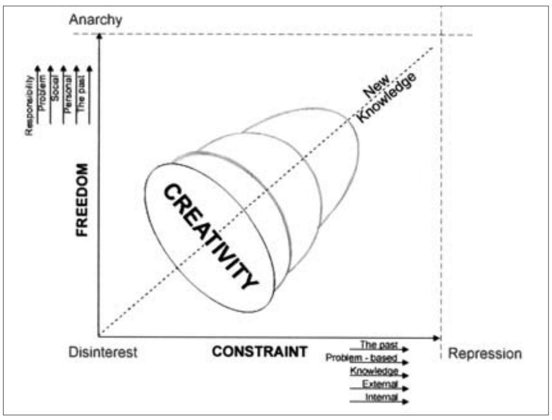
Now introduce a fixed point. What is the freedom of the circle (where on the plane can the centre get, so that the circle is a fixed (specified) distance from the fixed point? Here it is distance to the circle's circumference from the point that is being measured.

Negotiating and justifying conjectures with others may lead to modifying your first or second conjecture!

Now introduce a second, distinct fixed point. What freedom does the circle have (where can its centre be) if it is equidistant from the two fixed points? Again, your first conjecture may not include all the possibilities! Don't be limited by your intuitions! Try thinking of your circle moving so as to keep one of the distances constant while the distance to the other fixed point varies.

Now introduce a third fixed point distinct from the other two, and not on the line they determine. Again, what is the freedom of a circle which is equidistant from all three points?

Keep going until the circle runs out of freedom!



Constraints help us to focus on the areas of the problem and aid us in its solution; they give us the guidance needed to solve it. However, constraints also act as boundaries to our understanding: sometimes it is impossible to progress with too many barriers in our way.

The following two sets of exercises show how different constraints can lead to a focus on different features. In the second case we have found that different people respond in very different ways, some finding it boring, others finding that it opens new ways to see. In all of these, the task is to find the gradient of the lines joining the pairs of points:

Set 1	Set 2
(4,3) & (8,12)	(4,3) & (8,12)
(-2,-3) & (4,6)	(4,3) & (8,11)
(5,6) & (10,2)	(4,3) & (8,10)
(-3,4) & (8,-6)	(4,3) & (8,9)
(-5,3) & (2,3)	(4,3) & (8,8)
(2,1) & (2,9)	(4,3) & (8, a)
(p,q) & (r,s)	
(0,a) & (a,0)	

Can all constraints be circumvented? Let's dispense with the constraint of gravity – well, out goes the freedom to walk, to breathe, to live at all! So some constraints help us get going, while others might hinder progress. So where do the freedoms lie? At the beginning of a problem, the possibilities are infinite. But sometimes it feels as if there is no freedom at all, merely a pre-determined path, and this too is alright: one pleasure in mathematics is in nailing down a tiny piece of knowledge in a bewilderingly large field of exploration: it is a toehold in the infinite.

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Personal freedom/constraint

Developing as a mathematician involves an increasing readiness to take control of conditions, questioning the external ones we encounter and enjoying the freedom to impose and relax conditions on ourselves and others. We would like to see students wrestling with externally imposed conditions but also generating their own, perhaps in order to provide someone else with a puzzling challenge or to simplify an initially complex task – or maybe just to see what might happen if . . .

When doing mathematics each individual has internal constraints, such as confidence, previous knowledge, mood, learning style. Any or all of these can hinder or improve the experience. In the context of learning mathematics, we walk a slim beam between too much freedom (confusion, bewilderment, disinterest, fatigue, lack of clarity and certainty) and too much constraint (imposed authority, over-compartmentalisation, boredom, disinterest, fatigue, lack of link-making and internal understanding). Too much of either and the learner

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is likely to be bewildered or stymied. But if we get enough of both . . . there is space for learning. The reasons behind constraints are understood, links are made and we learn a little. We are empowered by our mathematical freedom and we move towards understanding the whole, not just a part.

There once was a quite sad bee
Who was very, very, very free.
It said "I certainly ain't
Under any constraint,
But from boredom I wish I could flee.

Another task which occupied some time was as follows:

$\frac{1}{2} - \frac{1}{3} = \frac{1}{6} \qquad \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \qquad \frac{1}{4} - \frac{1}{5} = \frac{1}{20}$

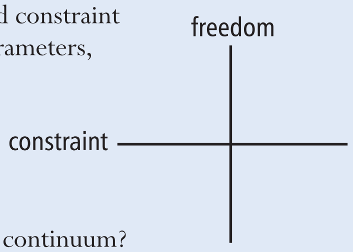
What is the same and what is different about these three statements, and so what is the class of examples of which they are exemplars? Characterise all pairs of unit fractions whose difference is a unit fraction.

If a problem has too much structure it might inhibit 'free thinking' and reward/encourage 'train spotting' with formulae being identified but the underpinning mathematics and justification ignored. On the other hand, if there is no structure the problem may be 'out of reach' for the audience and without some bridging, scaffolding or framing they are never going to be able to engage.

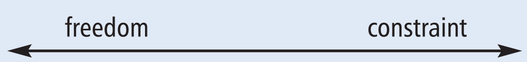
One way of viewing a learner's progress in mathematics is to view it as an evolution of understanding: it is made possible by the pressure of constraints but with freedom to try many possibilities; it is evidenced by growing fluency and access to larger classes of examples.

Different people will thrive best on different amounts of freedom and this may not be entirely related to past experiences. This also includes the freedom and constraints of personal knowledge. If you have no way of stepping into a problem because you do not know the underpinning mathematics then your knowledge is a constraint. However, sometimes too much knowledge can be a constraint;

Are freedom and constraint independent parameters, like two axes?



or are they on a continuum?



we might try to use algebra to solve a problem that would be better dealt with by iterative, intuitive, or other methods.

With freedom comes responsibility to use the freedom and not 'opt out'.

The national curriculum, which many see as a constraint on what they can teach, for others gives a structure within which they feel secure enough to explore. For students, it grants them the freedom of entitlement.

Contrasting freedom and constraint

Our . . .	gives us FREEDOM to . . .	and CONSTRAINS us . . .
ordinary language	generate and direct our thoughts so we can converse in the classroom	since there may be better, more precise, more conventional, less ambiguous words
maths symbols	communicate precisely and unambiguously, bypass language in thought, work with abstract ideas	as meanings are compressed; symbols may not exist for what I want to say; a symbolic culture can exclude others
culture/society	use tools relevant to the culture/society	by imposing a curriculum, a structure within which we are all expected to 'perform' and having social expectations
being human	exercise free will; and it is – human to be numerate – human to make patterns	as we persist in discerning/ imposing pattern; as we match our behaviour to situations
physical body	see, hear, write, do, gesticulate, feel	as we live within limited dimen sions; we have limited attention spans; we (mis)interpret what we sense

Freedom and constraint are like light and shade; though mutually opposing they make each other, and many other things, possible.

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And Malcolm Swan adds more on freedom and constraints

We all feel constrained: to produce exam results, keep up with the scheme of work, introduce the latest initiative, deliver impressive three part lessons... and so on. In turn, we constrain students by the tasks we give them, by the resources we do or don't provide, and by the social conventions we impose. Most constraints appear to be introduced to raise general standards and reduce diversity of

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Pace (often interpreted inaccurately as speed) is supposed to create interest and involvement.

experience, but they can also become stifling, lower expectations and become excuses for not using professional judgment.

Teachers respond to outside pressures differently. Some are obedient and follow the rules. Others do so more reluctantly, perceiving the conditions they work under as imprisoning. They become cynical, disillusioned and joyless. Yet others, the more confident and creative, playfully bend the constraints and use them imaginatively.

I offer just a few examples. I have recently been working with teachers of one-year repeat GCSE courses. These teachers frequently feel constrained to cover the entire syllabus again from the beginning and the effect is depressing:

> ... teachers say they are constrained by the amount of time allocated for GCSE lessons and speak about 'getting through the syllabus'. Often, teachers fail to take account of students' existing knowledge or the pace at which they learn ... All students, whether or not they have studied the subject before, receive the same tuition. In too many lessons, teachers do most of the work. Students are expected to sit and listen for long periods, to copy notes or take dictation. Frequently, they grow restless and lose concentration. [1] (para. 27)

The result of trying to cover everything too quickly is that students' learning needs are ignored. So why feel bound by the constraint of 'coverage'? What happens when we dare to challenge it? Here is a quote from one teacher at an FE college:

> *'One of the major pressures I feel is the obligation to cover everything in the GCSE specification and complete everything in the scheme of work. It is difficult to get through everything in under three hours a week. I recall a staffroom conversation in which we sounded like we were competing to see who had managed to 'cover' trigonometry in the shortest time possible. Is this effective teaching and learning? When I 'speed teach', I sometimes ask myself who is covering GCSE maths? Is it the students or is it just me? The interesting thing is that in the first year I missed great chunks out because I was ambling along so slowly. I rushed at the end. I had the majority (of students) and it was the best results the college has ever had.'*

The constraint to maintain pace is felt by many teachers and students. Pace (often interpreted inaccurately as speed) is supposed to create interest and involvement. While rapid questioning might have a role for developing fluency (where the goal is to be able to answer automatically without thought) it has, in my view, little place when developing understanding (where the goal is to share meanings and reflect more deeply). All our research into learning from mistakes and misconceptions (eg, Swan, 2001 [2]) underlines the need to take more time to listen to students, and encouraging students to listen to each other. This is a slow business and many teachers reject discussion because it simply takes too long. Teachers that care about learning should challenge the still(!) common practice of dividing book chapters by the number of weeks available to determine the 'pace of delivery'!

Then there is the constraint of writing 'lesson objectives' on the board and returning to them in the obligatory 'plenary' at the end of a lesson. While it seems obvious that well planned lessons will be more interesting and enjoyable than 'on the hoof', 'ad hoc' lessons, does this necessarily mean that lesson end points should be predetermined? I can't do better than to quote David Fielker: [3]

> *I am ... thinking of our professional space, which has been invaded by the aliens from the outer space of politics. We need freedom to make decisions in our classrooms:*
> - *freedom to leave things in the air when they cannot be resolved*
> - *freedom to change our goals when we can see they will not be achieved;*
> - *freedom to work on children's ideas instead of imposing our own, and therefore*
> - *the freedom to plan only the beginning of a lesson.*

Most teachers feel a powerful desire to 'tidy up' learning for students. When we feel constrained to 'sort out' problems for students or 'go through the answers' at the end of a lesson, we restore a sense of closure and completeness, but students then feel that they no longer need to think about the problem. While observing the FE teacher quoted above, I frequently noted his practice of leaving issues unresolved. When he left a group after an intervention, students would often discuss more intensively than before. At the end of a lesson the board would be covered with students' ideas (often incorrect), there would be no closure and students would leave still arguing about the mathematics. Resolutions, he would leave to a plenary discussion at the beginning of the next lesson.

I do apologise if this seems like a bit of a rant! Ultimately, I am arguing for a less passive acceptance of the constraints we all feel and a more critical, positive, creative reaction. Let's restore some professional judgment on such issues:

- Why *should* we attempt to cover the syllabus?
- Why *should* we write goals for lessons on the board?
- Why *should* all lessons proceed at a brisk pace?
- Why *should* all lessons conclude with a plenary?

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References

1 FEFC: ***GCSE in the Further Education Sector***. Coventry: Further Education Funding Council, 1997

2 M. Swan: *Dealing with misconceptions in mathematics*. In P. Gates (Ed.), ***Issues in mathematics teaching*** (pp. 147-165). London: RoutledgeFalmer, 2001

3 *D. Fielker: Shape, space and pace*. ***MT176***, September 2001, pp16-22

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