Playing with Dice

by **Colin Foster** and **David Martin**

Simple-sounding dice games can be surprisingly complicated to analyse. Let's throw an ordinary unbiased die and if it shows a square number then Colin gains a point and if it shows a non-square number then David gains a point. The winner is the first player to get 5 points. What is the probability that Colin will win this game?

The first thing to consider is that getting a square number and getting a non-square number are not equally-likely events. If we had equally-probable ways of gaining our points, then the situation would be completely symmetrical and there would be nothing to distinguish between us, so we would each have a probability of $\frac{1}{2}$ of winning. But in this scenario Colin and David have different probabilities (summing to 1) of gaining a point on each throw. What effect does this have?

Since there are only two square numbers (1 and 4) on an ordinary die, the probability p that Colin will gain a point is $\frac{2}{6}$, whereas the probability q that David will gain a point is $\frac{4}{6}$ (with $p + q = 1$). So clearly Colin is less likely to win this game than David is, since every time we throw the die Colin is at a disadvantage. But *how much* of a disadvantage is Colin suffering from? How could we find the probability that Colin will win the game? Is it simply $\frac{1}{3}$?

If the winner of the game were the first player to get 1 point, then $\frac{1}{3}$ would be the right answer. But instead the winner is the first player to get 5 points. Is this going to make a difference? To begin, let's suppose that the winner were the first player to get 2 points. How could Colin end up winning this game? There are two mutually-exclusive possibilities:

(i) Colin gets 2 points and David gets 0 points; and

(ii) Colin gets 2 points and David gets 1 point.

Case (i) has probability p^2, but with case (ii) there are two possible orders of gaining points, Colin–David–Colin or David–Colin–Colin, each with probability p^2q. (The possibility Colin–Colin–David is excluded because it reduces to scenario (i), since Colin would win before David gets to have a go.)

This gives a total probability:

P(Colin wins) $= p^2 + 2p^2q = p^2(1 + 2q)$.

Another way to arrive at this result is to consider the maximum number of throws required to determine the outcome, which is 3. Then we write

$(p + q)^3 = p^3 + 3p^2q + 3pq^2 + q^3$.

(Here we envisage continuing to throw until we have thrown three times, even though the outcome of the game may be known before the third throw.) The first two terms lead to a win for Colin, so his probability of winning is

$$\begin{aligned} p^3 + 3p^2q &= p^2\left(p + 3q\right) \\ &= p^2\left(p + q + 2q\right) \\ &= p^2\left(1 + 2q\right), \end{aligned}$$

as before. For our square/non-square example, substituting $p = \frac{1}{3}$ and $q = \frac{2}{3}$ gives

$$\text{P(Colin wins)} = \left(\frac{1}{3}\right)^2\left(1 + 2\left(\frac{2}{3}\right)\right) = \frac{7}{27} = 0.26$$

(correct to 2 decimal places). This is *less than* $\frac{1}{3}$.

Can you explain why?

Returning to the 'first to five' game, in which the winner is the first person to reach 5 points, it follows that if Colin achieves 5 points before David does, then David must obtain 0, 1, 2, 3 or 4 points. The final, winning go must be where Colin obtains his fifth square number, which he can do with probability p. So we can calculate Colin's probability of winning as

$$\left[p^4 + \binom{5}{1}p^4q + \binom{6}{2}p^4q^2 + \binom{7}{3}p^4q^3 + \binom{8}{4}p^4q^4\right]p,$$

where the five terms inside the brackets take account of the five different scores which David might obtain while Colin is obtaining his four square numbers, and the number of different ways in which David might obtain each of them. This can be written as

$$p^5\left[1+\binom{5}{1}q+\binom{6}{2}q^2+\binom{7}{3}q^3+\binom{8}{4}q^4\right].$$

Alternatively, by expanding $(p + q)^9$ we obtain Colin's probability of winning as

$$p^9+\binom{9}{1}p^8q+\binom{9}{2}p^7q^2+\binom{9}{3}p^6q^3+\binom{9}{4}p^5q^4,$$

which factorizes to

$$p^5\left[p^4+\binom{9}{1}p^3q+\binom{9}{2}p^2q^2+\binom{9}{3}pq^3+\binom{9}{4}q^4\right]$$

and substituting $p = \frac{1}{3}$ and $q = \frac{2}{3}$ into either of these gives

$$\text{P(Colin wins)} = \frac{2851}{19{,}683} = 0.14,$$ correct to 2 decimal places (see Note). So the probability that Colin will win this game is even lower. It seems that as the target number of points increases, Colin's chance of winning decreases. Why should this be?

It is interesting to observe why this second method is valid. It relies on there being only two possible winners, Colin or David, with the result that it is clear from the term in the expansion who has won, no matter in what order the points are scored. With 5 points to win, and the sum of the powers of p and q being 9, for Colin to win there must be a power of p of 5 or more, which means that the power of q will be 4 or less, so David could not have won.

It is also interesting to explore how the probability that Colin will win varies as the values of p and q change ($p + q = 1$). Some possible values for the 'first to five' game are shown in Table 1, and the relationship is graphed in Figure 1. The graph is by no means a straight line from the origin to (1, 1). Instead, deviations at very small or very large values of p are small, whereas small deviations from $\frac{1}{2}$ have a much larger effect. This means that even a *slightly* biased die could have a noticeable effect on what might appear to be a perfectly fair game.

Table 1 The probability of Colin winning the 'first to five' game for different values of p

p	Probability that Colin wins (correct to 4 decimal places)
0.1	0.0009
0.2	0.0196
0.3	0.0988
0.4	0.2666
0.5	0.5000
0.6	0.7334
0.7	0.9012
0.8	0.9804
0.9	0.9991

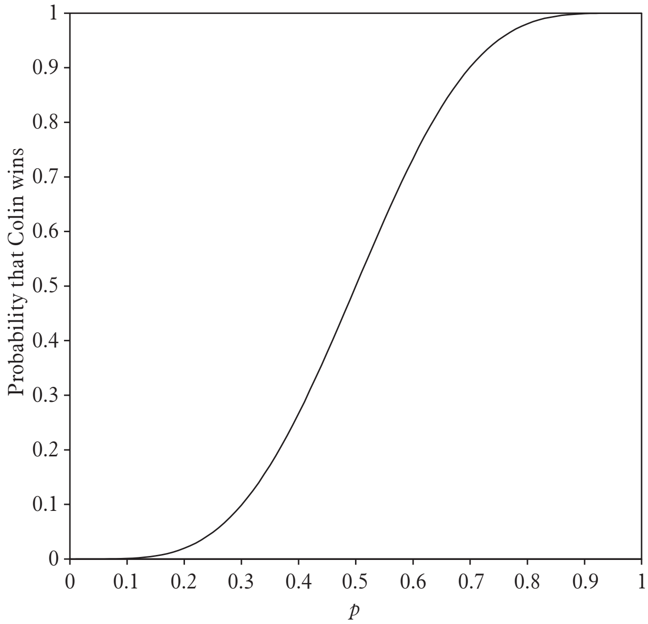

Fig. 1 The probability of Colin winning the 'first to five' game as p varies

Note

In comparing these two solutions, it is interesting to consider why

$$1+\binom{5}{1}q+\binom{6}{2}q^2+\binom{7}{3}q^3+\binom{8}{4}q^4$$

$$=p^4+\binom{9}{1}p^3q+\binom{9}{2}p^2q^2+\binom{9}{3}pq^3+\binom{9}{4}q^4.$$

We note that $(p + q)^n = 1$ and multiply each of the terms in the first by, in turn, $(p + q)^4$, $(p + q)^3$, $(p + q)^2$, $(p + q)$ and 1, and expand and collect terms in powers of p and q to obtain the second (this is not for the faint-hearted!). The coefficients of these two corresponding expressions can be found in Pascal's triangle as a diagonal (of pentatope numbers) and a row. Students may like to think further about how these two solutions interrelate and relate to entries in Pascal's Triangle.

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Authors Colin Foster, School of Education, University of Nottingham, Jubilee Campus, Wollaton Road, Nottingham NG8 1BB.
e-mail: **c@foster77.co.uk**
website: **www.foster77.co.uk**
David Martin, School of Education, University of Nottingham, Jubilee Campus, Wollaton Road, Nottingham NG8 1BB.
e-mail: **d.martin@virginmedia.com**