Communications

Peripheral mathematical knowledge

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Zazkis and Mamolo (2011) explore the benefits of a teacher employing subject-matter knowledge which is close to their own mathematical horizon. Although this knowledge is well beyond what the learner is seeking to master, Zazkis and Mamolo present a convincing case for its importance in influencing a teacher's choices in the classroom. They define *knowledge at the mathematical horizon* as undergraduate mathematics or equivalent that has some bearing on the mathematics the learners are doing. As Zazkis and Mamolo explain, "The horizon being 'farther away' for the teacher enables him or her to see more features and attributes of an object, and to gain a more in-depth appreciation for what exists in the outer world" (p. 10). The model which Zazkis and Mamolo outline might be illustrated as in Figure 1, with a progression upwards indicating greater mathematical sophistication. Where the teacher's advanced mathematical knowledge meets the learner's school mathematics, there is fruitful interaction.

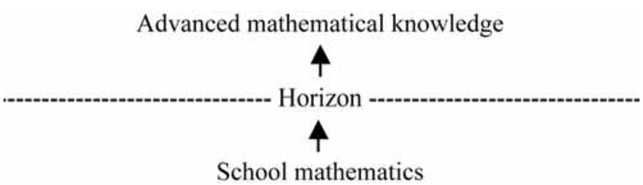

Figure 1. Knowledge at the mathematical horizon.

It seems to me, however, that there is another kind of mathematical knowledge which a teacher may possess. This knowledge is useful pedagogically but is *not* knowledge which the learner might be expected to learn now, or even (unless they become a mathematics teacher themselves) in the future. It may or may not be currently "beyond" the learner (in terms of difficulty), but it is different in kind from the mathematics that they would be expected to learn. I will

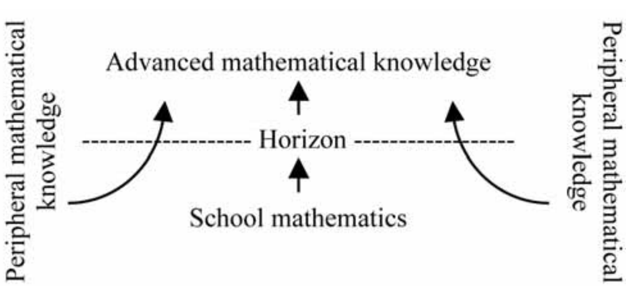

Figure 2. Peripheral mathematical knowledge.

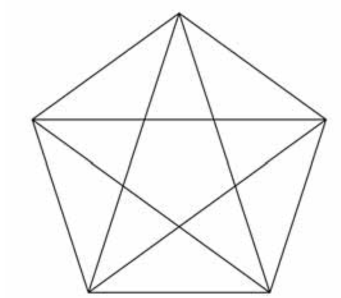

Figure 3. How many triangles? (Zazkis & Mamolo, 2011, p. 8)

refer to this knowledge as *peripheral mathematical knowledge* and will give some examples below. Peripheral mathematical knowledge can exist anywhere along the vertical axis of "difficulty", and I envision it coming in from the sides, cushioning and supporting the learner's mathematical trajectory upwards (Figure 2). Significantly, I would probably *not* regard this knowledge as an important part of the learner's mathematical journey, although it is of great value to the teacher, as I will attempt to illustrate.

Zazkis and Mamolo (2011) begin their article with the question shown in Figure 3. As I read, I responded by saying "35", since the problem is familiar to me. This instant recollection of a result would be a simple example of peripheral mathematical knowledge (mathematical trivia, in this case), which is useful for a classroom teacher but not, perhaps, for anyone else. I would not expect a professional mathematician to know it, nor would I regard it as an important fact for learners to know. However, it can be convenient for a mathematics teacher.

There are many mathematical facts that I did not know before I began teaching and which I have picked up "on the job". For example, I know that there are 11 nets of a cube, and I am glad that I know it. First, it is helpful to know that there are sufficiently few for it to be worthwhile asking learners to find them all. Second, although when undertaking such a task the value for learners is not in the final answer but in the spatial thinking and reasoning involved in systematically tackling the cases, knowing that there will be 11 nets helps the teacher to manage the process more powerfully.

Such knowledge is more than just the memory of problems I have done before. I probably did these tasks myself at school or since but, seeing little significance then in the final answers, I subsequently forgot them. For me as a learner of mathematics, this in no way diminished the value of the experience. It was only when I began teaching mathematics that such results became significant for me and only a mathematics teacher, I suggest, would see much value in them. Peripheral mathematical knowledge is mathematical rather than pedagogical, but it can be thought of as an applied mathematics where the application is teaching. Just as an engineer might study their pure mathematics differently from the way in which a pure mathematician might, so a mathematics teacher might find different points of interest in a piece of mathematics as a consequence of being a teacher.

Examples of peripheral mathematical knowledge will inevitably vary from teacher to teacher and with the age of learners they work with. Below is a selection of some of my

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own peripheral mathematical knowledge. In each case, I would not care if learners did not have that knowledge (either now or at any time in their future mathematical journey), but, as a teacher of mathematics, *I* find it useful to know that:

- a cube has 9 planes of symmetry
- all planar quadrilaterals tessellate in 2D and tetrahedrons do not tessellate in 3D
- both $x^2 + 17x + 30$ and $2x^2 + 17x + 30$ factorise (and how to generate other such pairs)
- $\frac{19}{95}$, $\frac{16}{64}$, $\frac{49}{98}$ and $\frac{26}{65}$ (and their reciprocals) are the only 2-digit over 2-digit fractions that give equivalent fractions by cancelling off identical digits
- $2^{10} = 1024$ without checking
- tan 35° is very close to 0.7

I claim that whereas the process of *coming to know* these things may be of great value for learners, *knowing them* may not be. Yet this is not because such knowledge is in any sense beyond them. On the contrary, such knowledge is never likely to assume much importance for them unless they teach mathematics themselves. It is not over their horizon; it is outside of their peripheral vision.

Peripheral mathematical knowledge encompasses more than isolated facts and figures; knowing how to draw 2D shapes that have any order of rotational symmetry but no lines of symmetry, for instance, or knowing how to make up equations of the form:

$$\frac{ax+b}{cx+d} = \frac{ex+f}{gx+h},$$

(with integers for a to h) which lead to quadratic equations that factorise, might be included, and readers will be able to think of many more.

Mrs White's conjecture

When I read in Zazkis and Mamolo's (2011) article that Mrs White knew that the answer to the question shown here in Figure 3 had to be a multiple of 5, my immediate reaction was "surely not"; it just felt too simple. Does this response represent some kind of mathematical knowledge? My experience with counting things in symmetrical arrangements has led me to expect complexity, and the idea that five-fold symmetry implies that the number of triangles must be a multiple of 5 seemed too easy. Sometimes, when a learner offers me a conjecture which I do not know to be false, I have the feeling "if that were true, I would know it already." Is a "gut reaction" or a feeling of unease a kind of mathematical knowledge? How do such feelings develop? (The answer "by experience" does not tell me much.)

I continued by thinking about a triangle and a square (Figure 4). The square contains 8 triangles, which is a multiple of 4 (the number of sides), but a triangle is just 1 triangle, so not a multiple of 3. This early exploration made me doubt the conjecture. However, I am familiar with the way in which some sequences "don't work for the first one", so I did not get too excited. What is going on here seems to be cautiousness – a reluctance to assume that Mrs White is wrong, yet a feeling that she may be. My decision to test the conjecture in simpler cases and my awareness of exceptional first terms in sequences feel valuable, but where do they come from?

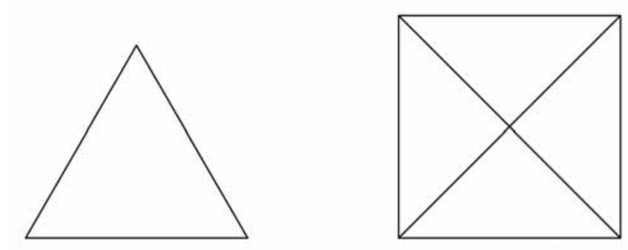

Figure 4. Simpler examples.

The process I was going through reflects a common situation for me in the classroom. When something is said and I do not know whether or not it is correct, I seek to behave open-mindedly and mathematically in the way that I respond. In this case, I drew examples with 6 and 7 sides (Figure 5) but found that there were so many triangles that it was too difficult to count them all and be sure that I had not omitted any. My drawings were useful, however, since I noticed that sometimes more than two lines pass through a point. As a result, I felt that some potential triangles would be lost, depending on the symmetry, and that any simple formula would therefore be unlikely to work in all cases.

The general problem turns out to be difficult (Sommars & Sommars, 1998) and, in general, the number of triangles is *not* a multiple of the number of sides (see Table 1). 35 *is* a multiple of 5, however, which makes me wonder whether Mrs White is seeing something that I am missing or has an intuition that I do not have. Perhaps Mrs White said what she said for a pentagon but would not have said it for a hexagon. This suggests to me that what a teacher "knows" in the classroom is a personal and complex matter.

Conclusion

Within a traditional transmission-teaching paradigm, the teacher's subject-matter knowledge is identical in kind to the knowledge which learners are acquiring (or seeking to acquire): the teacher may be further down the road, but it is the same road. But in classrooms in which learners are viewed as constructing their own knowledge of mathematics through personal exploration, the subject knowledge required by the teacher is more complex and multifaceted. The subject-matter knowledge that helps a teacher to teach mathematics is more than simply what the learners will be learning later on. In every case in Zazkis and Mamolo's (2011) article, the teacher's horizon knowledge could be regarded as mainstream mathematical knowledge that anyone studying more mathematics for any purpose might be likely to learn if they go far

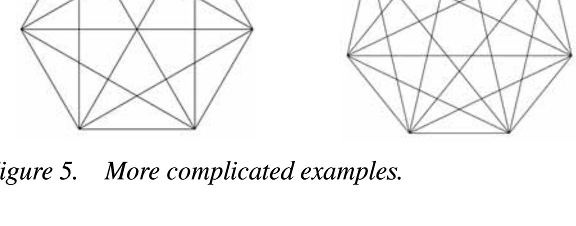

Figure 5. More complicated examples.

n	Number of triangles	Is the number of triangles a multiple of *n*?
3	1	NO
4	8	yes
5	35	yes
6	110	NO
7	287	yes
8	632	yes
9	1302	NO
10	2400	yes
11	4257	yes
12	6956	NO
13	11 297	yes
14	17 234	yes
15	25 935	yes
16	37 424	yes
17	53 516	yes
18	73 404	yes
19	101 745	yes
20	136 200	yes
21	181 279	NO

Table 1. The number of triangles in a regular n-gon in which all the diagonals are drawn; situations in which the number of triangles is not a multiple of the number of sides are shaded [1].

enough. There is no reason to believe that the results referred to would be of more use to a mathematics teacher than to any other user of mathematics. By contrast, the peripheral mathematical knowledge that I am seeking to describe lies on the verges of this main highway, yet is no less important for that.

Shulman (1987) argued that pedagogical content knowledge:

> represents a blending of content and pedagogy into an understanding of how particular topics, problems, or issues are organized, represented, and adapted to the diverse interests and abilities of learners, and presented for instruction. Pedagogical content knowledge is the category most likely to distinguish the understanding of the content specialist from that of the pedagogue. (p. 8)

It is perfectly possible to agree with Shulman's final sentence while also asserting that differing subject-matter knowledge may also be an important distinguishing feature.

Note

[1] See the On-Line Encyclopedia of Integer Sequences (http://oeis.org), sequence A006600.

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Teachers' advanced mathematical knowledge for solving mathematics teaching challenges: a response to Zazkis and Mamolo

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Like Zazkis and Mamolo (2011), we uphold the premise that a solid knowledge of advanced mathematics is needed for effective teaching of mathematics. With that in mind, we are interested in discussing the nature of horizon content knowledge, as used within the mathematical knowledge for teaching framework (Ball, Thames & Phelps, 2008).

Mathematical knowledge for teaching (MKT) continues to generate a huge number of papers and all kinds of praise and criticism in scientific settings. Far from adding to this work, our aim is to rescue the concept of horizon content knowledge and re-conceptualise it. We wish to highlight a fundamental premise underlying the MKT framework: teachers' mathematical knowledge belongs to their professional knowledge, and thus has to do with, and cannot be separated from, the teaching challenges that they approach in their practice (Stylanides & Stylanides, 2010). Our critique of Zazkis and Mamolo's paper is much more in terms of their assumptions about the nature of the mathematical knowledge that elementary and secondary teachers need, rather than in terms of their conceptualization of knowledge at the mathematical horizon.

Zazkis and Mamolo use several examples to illustrate how certain knowledge of what they consider to be advanced mathematics is used by teachers to deal with classroom situations. In the first one, the teacher asks the students to calculate the number of triangles in a regular convex pentagon with all diagonals drawn in. Some knowledge of symmetry helps her to see that the number of triangles has to be a multiple of five and to solve the mathematical problem. With this solution in mind, she directs her pupils to solve the problem using symmetries. It is clear that her advanced knowledge on this topic permits her to deal with a teaching situation in a very elegant way. This example reinforces the premise that advanced mathematics is the best - even the essential - background for teaching mathematics. However, after reading Zazkis and Mamolo's description of the situation, we doubt whether this way of using advanced knowledge, and probably the way in which that knowledge was acquired, allows teachers to build on students' knowledge or to interpret alternative solution paths implicit in students' answers. Mathematical problems like that of counting triangles behave very differently in a pure mathematical setting than in an educational context. They become much more complex in an educational context because, among other things, they necessarily involve the mathematical reasoning of the people we have the responsibility to teach.

There was probably no way for these 8-9 year old students to understand why the teacher directed them to identify different kinds of triangles, and then look for five triangles of each kind. Moreover, this approach may reinforce the preconception that problem solving necessarily calls for some brilliant idea, without which the solution remains utterly unattainable.

About the kind of mathematical knowledge that teachers need

Zazkis and Mamolo lead us to conclude that advanced mathematical knowledge is a necessary tool for the teachers to solve, in the classroom, the problems they pose. We certainly agree, but to us it feels like another case of those existence theorems that leave us longing for an explicit construction procedure. We maintain that another perspective is possible, one that also considers, and even advances, the knowledge of the mathematics education community. Let us go back again to one of the examples used in the paper: in Example 3 they describe how the teacher's knowledge about group theory sparks her insight to interpret several confusions and errors concerning the reciprocal and the inverse of a function in terms of a misgeneralization of previous work with negative exponents. Zazkis and Mamolo's description of the teaching situation speaks very well about this (fictional) teacher, and also about the advanced mathematical knowledge the teacher seems to have, but very badly about her mathematics education teachers. Students' confusion about $1/f(x)$ and $f^{-1}(x)$ is well known and the teacher should have heard about it in any course about teaching analysis. The interesting question for us, which is again a mathematics education problem, is what kind of solid education in analysis and group theory the teacher should have received in order to avoid the genesis of this misunderstanding. Perhaps she would not have been surprised about her students' confusion if she had been guided to reflect on the structure of the set of functions under composition immediately after the study of the multiplication of functions, trying to understand why the properties of multiplication in **Q** or **R** do not hold for general functions. In any case, giving advanced mathematical knowledge to teachers without taking into account its relevance for teaching practice is like providing a carpenter with a new, unknown tool without any information about how it can facilitate her work. Surely, with observation and reflection she will be able to elucidate some aspects of its uses and possible potentialities, but her professional problems are different.

Moreover, advanced mathematical knowledge is not meant to be directly applied in teaching situations, but instead is an essential ingredient for a deep understanding of basic mathematics, to an extent not usually covered in the syllabus of many mathematics faculties. To better explain what we intend to say, we use an example drawn from our own research, in which students had just started a unit on equivalent fractions:

> Mr. Paulino explains the idea of equivalent fractions. He takes a piece of paper, folds it twice in half and colors one of the rectangles obtained. He unfolds it and says: "We have colored one quarter of the paper". The students nod patiently. Afterwards, Mr Paulino folds the same piece of paper three times and, when unfolding it, he says: "The fraction colored is now two eighths. One quarter and two eighths are equivalent fractions because they represent the same quantity." He writes on the board:
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> and says: "Notice that 1 times 8 is 8, and 4 times 2 is 8. One quarter and two eighths are equivalent fractions because their crossed product is equal." He continues by saying: "The second fraction is obtained by multiplying both numerator and denominator of the first one by 2."

By folding a piece of paper, he implicitly defines a fraction as a part of a whole. Immediately afterwards, he refers to a fraction as the division between two numbers when he asserts that equivalent fractions represent the same quantity. The teacher knows different ways to define a fraction from his university studies and, in this point, we agree with Zazkis and Mamolo, but we consider that deeper reflection is needed.

There is an intraconceptual connection (inner horizon, using Zazkis and Mamolo's terminology) between these two meanings of a fraction which is not trivial for the students. This reflection is crucial for the mathematics teacher and, perhaps, not so much for others using mathematics professionally. The connection is not made explicit by the teacher, who freely moves between these two meanings and leaves the students to their own devices in the process of giving them coherence. Moreover, the definition of equivalent fractions is supposed to be generalised from one particular case to every pair of equivalent fractions. Immediately afterwards, the teacher writes down the same fraction and reduces to an observation what is usually taken as the definition of the equivalence relation in the field of fractions of **Z**: two ordered pairs of integers (a, b) and (c, d), with positive b and d, are equivalent if $a \cdot d = b \cdot c$. It is introduced as an almost mnemonic rule and it does not connect with the meaning of a fraction as a part of a whole. The "rule" is meaningful only once the students are familiar with the multiplication of fractions.

Moreover, the teacher's introduction to equivalent fractions shown in this episode ends by explaining a procedure to obtain equivalent fractions, namely, by multiplying both the numerator and denominator of the first one by 2. This procedure of generating equivalent fractions is normally presented using only integer multiples and produces a foundational misunderstanding for the students: they assume equivalent fractions to be characterized by one of them being the result of multiplying/dividing the numerator and denominator of the other by the same integer, which is not the most general operation possible (for instance, 4/6 and 6/9 are equivalent). If this is considered advanced mathematical knowledge, it is certainly not the focus of general university mathematics.

About the theoretical approach to horizon content knowledge

The notion of horizon content knowledge is given by Zazkis and Mamolo in terms of the application of the notion of "advanced mathematical knowledge", which corresponds to the "knowledge of the subject matter acquired during undergraduate studies at colleges or universities" (Zazkis & Leikin, 2010 p. 264). The notion is therefore grounded in the power of an institution and those who work there. This kind of approach leaves very little space for a deep intellectual debate about how we can understand the problems of mathematics education. We emphasize that our professional task of teaching mathematics to primary and secondary students, as well as to future elementary and secondary school teachers, requires a much broader perspective on the nature of knowledge.

We mentioned at the beginning of this communication that our purpose is to explore the conceptualization of horizon content knowledge. Zazkis and Mamolo's description in terms of inner and outer horizon is very stimulating and permits us to refine our own approach, which conceptualizes horizon content knowledge in terms of connections between mathematical concepts and ideas, grounded in the coherence of mathematics, in which all concepts and ideas are precisely defined and logically interwoven.

Mathematical content knowledge cannot be solid without connections, and this leads us to think about horizon content knowledge as a key necessary prerequisite of mathematical knowledge for teaching. However, after analyzing Zazkis and Mamolo's paper, we have the feeling that they articulate all their reflection around the premise that mathematical teaching problems, and thus theoretical outcomes in the field of mathematics education, should be subordinated to the problem of teachers' learning of advanced mathematics. We have focused our response on discussion of this aspect, emphasizing the need for teachers to construct deep knowledge of the connections within mathematical content as a basis to enhance students' learning of mathematical structure (Vale, McAndrew & Krishnan, 2011). We hope to have further opportunities to think together about the conceptualization of horizon content knowledge, and thus on its impact on practice and training.

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Apologies to Marc Schäfer, whose name was mis-spelt in several places in the *Communications* section of the last issue.

The selected quotations in this issue commemorate the life and work of Martin Hughes (1949-2011). *Children and Number,* originally published in 1986, was reprinted at least twelve times.