Foster, C. (2014). Equal volumes of revolution.
Mathematical Gazette, 98(543), 504–507.

98.31 Equal volumes of revolution

Consider the region R in the xy-plane bounded above by the line $y = 2x$, below by the x-axis and to the right by the line $x = 1$. It may be observed that the volume of revolution V_x of the region R about the x-axis (Figure 1a) is equal to the volume of revolution V_y of the same region R about the y-axis (Figure 1b):

$$V_x = \int_0^1 \pi y^2\, dx = \int_0^1 \pi (2x)^2\, dx = \frac{4\pi}{3},$$

$$V_y = \pi 1^2.2 - \int_0^2 \pi x^2\, dy = 2\pi - \int_0^2 \pi \left(\frac{y}{2}\right)^2 dy = 2\pi - \frac{2\pi}{3} = \frac{4\pi}{3}.$$

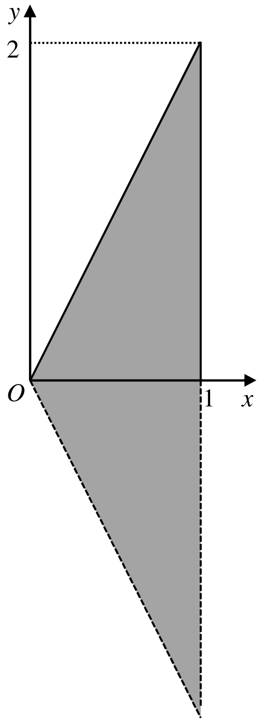

FIGURE 1a

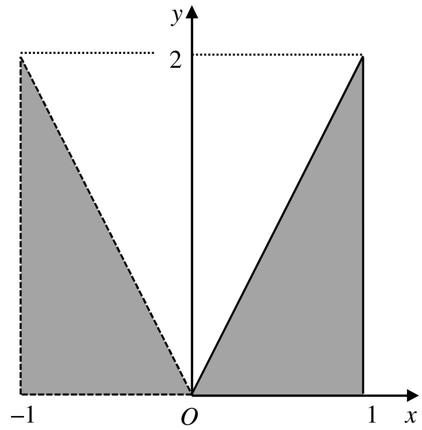

FIGURE 1b

This observation provokes the question whether there are other regions in the xy-plane with the property that $V_x = V_y$, an investigation that may be accessible and instructive for a competent sixth-form student.

It is possible to see straightaway that any region in the first quadrant which is symmetrical about the line $y = x$ should have this property, since the solids of revolution generated about the two axes would be identical and thus trivially have the same volume. For example, the unit square with vertices at (0, 0), (1, 0), (1, 1) and (0, 1) would clearly have the same volume of revolution about either axis, as would the triangular region contained between the axes and the line $y = a - x$, where $a > 0$. Another example would be the quadrant contained between the axes and the curve $y = \sqrt{1 - x^2}$. In all of these cases, the volumes of revolution are trivially equal, because the same solid is generated by rotating the region about either axis. So let us look for more regions like the region R above which are *not* symmetrical about the line $y = x$.

To calculate the volume of revolution V about the y-axis of a region under the curve $y = f(x)$ between $x = a$ and $x = b$, we notice that it is advantageous for our purposes to use the less well-known (at least in school) formula:

$$V = \int_a^b 2\pi xy \, dx$$

in which we integrate with respect to x, rather than the more usual approach in which we integrate with respect to y. In our formula, we take elements of volume which are cylindrical shells with radius x, height y and thickness dx (Figure 2).

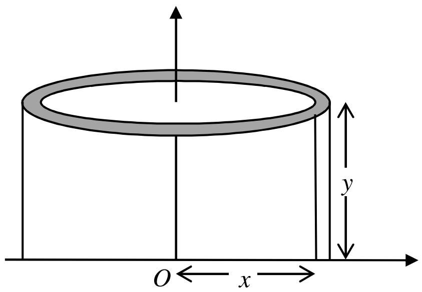

FIGURE 2

Without loss of generality, we may choose to operate on the closed interval [0, 1] on the x-axis and define our region R as bounded above by the curve $y = f(x)$, below by the x-axis, to the left by the line $x = 0$ and to the right by the line $x = 1$. To avoid trouble, we shall assume $y \geqslant 0$ for all x on [0, 1]. Thus we may express our condition that $V_x = V_y$ as

$$V_x - V_y = \int_0^1 \pi y^2 \, dx - \int_0^1 2\pi xy \, dx = 0.$$

Since the two integrals have the same limits, we may rewrite the condition as

$$\pi \int_0^1 y(y - 2x) \, dx = 0, \qquad (*)$$

where the π at the front may be cancelled, since it is non-zero.

Clearly one way in which this integral can be zero is if the integrand is identically zero; in other words if $y(y - 2x) = 0$, which can happen if, and only if, either $y = 0$ (which results in no region) or $y = 2x$, which was the example with which we began.

However, since for a suitable choice of $y \geqslant 0$ the integrand in $(*)$ may nevertheless be negative for some values of x, we expect that there will be other functions $y = f(x)$ which will satisfy this equation. Let us attempt to find some. Suppose, as a trial solution, that we let $y = ax^k$, where $a > 0$ and $k \geqslant 0$, ensuring that y is positive and finite for all x on $[0, 1]$. Then y will be a solution to our problem if, and only if, there are values of a and k which satisfy

$$\int_0^1 ax^k \left(ax^k - 2x\right) dx = 0.$$

Simplifying,

$$a \int_0^1 \left(ax^{2k} - 2x^{k+1}\right) dx = 0,$$

and cancelling the $a\,(\neq 0)$, we have

$$\left[\frac{ax^{2k+1}}{2k+1} - \frac{2x^{k+2}}{k+2}\right]_0^1 = 0,$$

giving

$$\frac{a}{2k+1} - \frac{2}{k+2} = 0.$$

Since $k \geqslant 0$, we can write

$$a = \frac{2(2k+1)}{k+2},$$

which is always positive.

Hence, we have an infinity of solutions of the form $y = ax^k$. Here are some examples:

- when $k = 0, a = 1$, and we obtain the line $y = 1$, making the region R the unit square with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$ and $(0, 1)$ that we found earlier;
- when $k = 1$, $a = 2$, giving $y = 2x$, the solution with which we began;
- when $k = 2, a = 5/2$, giving the parabola $y = 5x^2/2$;
- when $k = 3$, giving the cubic $y = 14x^3/5$.

Clearly it is straightforward to generate as many solutions of this form as we wish. These functions are all positive for $x > 0$, and substitution and integration verifies that these are indeed solutions to the original problem.

Students might profitably explore other classes of solutions and gain much useful practice at integration, algebraic manipulation and curve sketching in the course of this pursuit.

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