93.26 Isometric graphs

During a desperate shortage of squared paper, mathematics must nevertheless go on, and so mathematicians have to resort to isometric paper instead, on which to draw their graphs. Instead of the usual orthogonal x and y axes, they draw X and Y axes at $\frac{\pi}{3}$ and plot the coordinates at the isometric lattice points in the plane. How do familiar graphs look when plotted on isometric axes?

From Figure 1, we can relate a point (x, y) in the left drawing to the point (X, Y) in the right drawing by $x = X\cos\frac{\pi}{6}$ and $y = Y + X\sin\frac{\pi}{6}$. So, in general, the point with ordinary cartesian coordinates (x, y) has coordinates $\left(\frac{2\sqrt{3}}{3}x, y - \frac{\sqrt{3}}{3}x\right)$ with respect to isometric axes. Alternatively, the point referred to as (X, Y) in the isometric plane is $\left(\frac{\sqrt{3}}{2}X, \frac{1}{2}X + Y\right)$ relative to orthogonal cartesian axes. (These equations enable any of the 'isometric graphs' below to be produced using ordinary graph-drawing software by entering the appropriately transformed equations.)

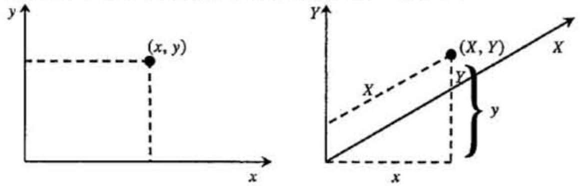


FIGURE 1

So we can investigate the properties of working in 'isometric land' and consider how the graphs of familiar equations such as $Y = X^2$ will look when plotted in this way.

1. *Straight lines*

Straight lines are still straight, since we can write

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \frac{2\sqrt{3}}{3} & 0 \\ -\frac{\sqrt{3}}{3} & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

and straight lines remain as straight lines under linear transformations.

If a straight line has equation $Y = mX + c$ in the isometric plane (where m is the gradient and c is the Y intercept), then $y = \frac{\sqrt{3}}{3}(2m + 1)x + c$ is the equation of the same line in the cartesian plane. The gradient m' of such a line, as usually defined (relative to horizontal and vertical directions), is therefore $m' = \frac{\sqrt{3}}{3}(2m + 1)$ so, as m increases, m' increases at the rate $\frac{dm'}{dm} = \frac{2\sqrt{3}}{3}$, which is greater than 1, so the gradients of lines increase more quickly on the isometric axes than they do on cartesian. Lines parallel to the Y-axis have equations $X = k$ (or $x = \frac{\sqrt{3}}{2}k$ if you prefer).

2. *Parabolas*

Conics will stay as conics, since the transformation is linear.

Beginning with $Y = X^2$, we obtain $y = \dfrac{4x^2 + \sqrt{3}x}{3} = \dfrac{4}{3}\left(x + \dfrac{\sqrt{3}}{8}\right)^2 - \dfrac{1}{16}$, so the transformed curve is still a parabola, but it is *not* symmetrical about the Y-axis, having a line of symmetry at $x = -\frac{\sqrt{3}}{8}$ or $X = -\frac{1}{4}$, so the minimum point is $\left(-\frac{\sqrt{3}}{8}, -\frac{1}{16}\right)_{\text{cartesian}}$ or $\left(-\frac{1}{4}, \frac{1}{16}\right)_{\text{isometric}}$. (In the isometric plane, we take 'minimum point' to mean the position at which there is a *horizontal* tangent, rather than a tangent parallel to the X-axis, which, of course, happens at the origin, as in the cartesian representation.)

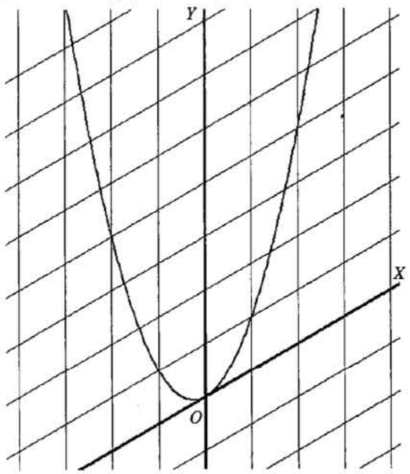


FIGURE 2

3. *Circles*

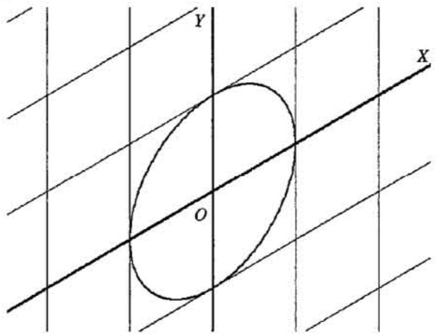


FIGURE 3

The unit circle centred on the origin, $X^2 + Y^2 = 1$, transforms to the ellipse $5x^2 - 2\sqrt{3}xy + 3y^2 = 3$ which, by symmetry, has its major axis along the line $y = \sqrt{3}x$ and its minor axis along $y = -\frac{\sqrt{3}}{3}x$. Solving the equations of each of these lines separately with the equation of the ellipse gives the coordinates $\pm\left(\frac{\sqrt{6}}{4}, \frac{3\sqrt{2}}{4}\right)$ and $\pm\left(\frac{\sqrt{6}}{4}, -\frac{\sqrt{2}}{4}\right)$, from which we obtain a semi-major axis length of $\frac{\sqrt{6}}{2}$ and a semi-minor axis length of $\frac{\sqrt{2}}{2}$; hence, the eccentricity is $\frac{\sqrt{6}}{3}$. (These values are all given as viewed in the x-y plane.) Figure 4 below shows the ellipse, together with the circles $x^2 + y^2 = \frac{1}{2}$ and $x^2 + y^2 = \frac{3}{2}$, which are tangents at the ends of the minor and major axes respectively.

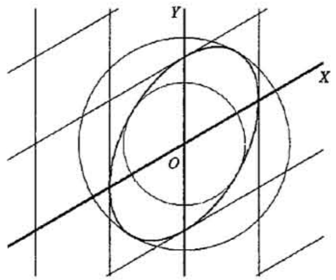


FIGURE 4

4. *Exponentials*

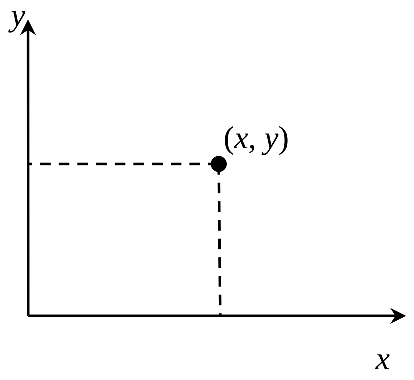
One question is whether the graph of an equation such as $y = 2^x$ climbs more quickly or less quickly when drawn in the isometric way. Converting $Y = a^X$, where a is a constant greater than zero, into $y = a^{2\sqrt{3}x/3} + \frac{\sqrt{3}}{3}x$ and differentiating gives $\frac{dy}{dx} = \frac{2\sqrt{3}\ln a}{3} a^{\frac{2\sqrt{3}}{3}x} + \frac{\sqrt{3}}{3}$, whereas the derivative of $y = a^x$ is merely $\frac{dy}{dx} = a^x \ln a$. The difference in gradient when $x = 0$ (for instance) is therefore $\frac{\sqrt{3}}{3}\left((2 - \sqrt{3})\ln a + 1\right)$, which is greater than zero when $a > e^{-(2+\sqrt{3})}$. So a graph such as $y = 2^x$ does indeed grow faster on isometric axes.

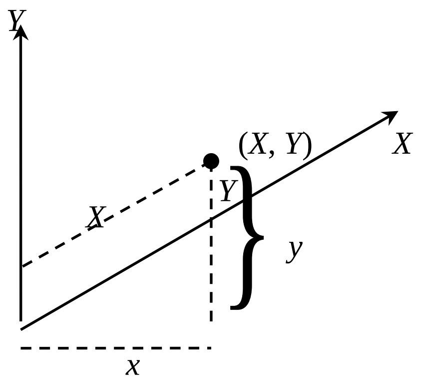
COLIN FOSTER
King Henry VIII School, Warwick Road, Coventry CV3 6AQ
e-mail: *colin@foster77.co.uk*

Isometric Graphs

COLIN FOSTER

During a desperate shortage of squared paper, mathematics must nevertheless go on, and so mathematicians have to resort to isometric paper instead on which to draw their graphs. Instead of the usual orthogonal x and y axes, they draw X and Y axes at $\frac{\pi}{3}$ and plot the coordinates at the isometric lattice points in the plane. How do familiar graphs look when plotted on isometric axes?





From the diagrams, we can relate a point (x, y) in the left drawing to the point (X, Y) in the right drawing by $y = Y + X\sin\frac{\pi}{6}$ and $x = X\cos\frac{\pi}{6}$. So, in general, the point with ordinary Cartesian coordinates (x, y) has coordinates $(\frac{2\sqrt{3}x}{3}, y - \frac{\sqrt{3}x}{3})$ with respect to isometric axes. Alternatively, the point referred to as (X, Y) in the isometric plane is $(\frac{\sqrt{3}X}{2}, \frac{X}{2} + Y)$ relative to orthogonal Cartesian axes. (These equations enable any of the ‘isometric graphs’ below to be produced using ordinary graph-drawing software by entering the appropriately transformed equations.)

So we can investigate the properties of working in ‘isometric land’ and consider how the graphs of familiar equations such as $Y = X^2$ will look when plotted in this way.

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and straight lines remain as straight lines under linear transformations.

If a straight line has equation $Y = mX + c$ in the isometric plane (where m is the gradient and c is the y-intercept), then $y = \frac{\sqrt{3}}{3}(2m+1)x + c$ is the equation of the same line in the Cartesian plane. The gradient m' of such a line, as normally defined (relative to horizontal and vertical directions), is therefore $m' = \frac{\sqrt{3}}{3}(2m+1)$, so, as m increases, m' increases at the rate $\frac{\mathrm{d}m'}{\mathrm{d}m} = \frac{2\sqrt{3}}{3}$, which is greater than 1, so the gradients of lines increase more quickly on the isometric axes than they do on Cartesian. Lines parallel to the Y-axis have equations $X = k$ or $x = \frac{\sqrt{3}k}{2}$ if you prefer.

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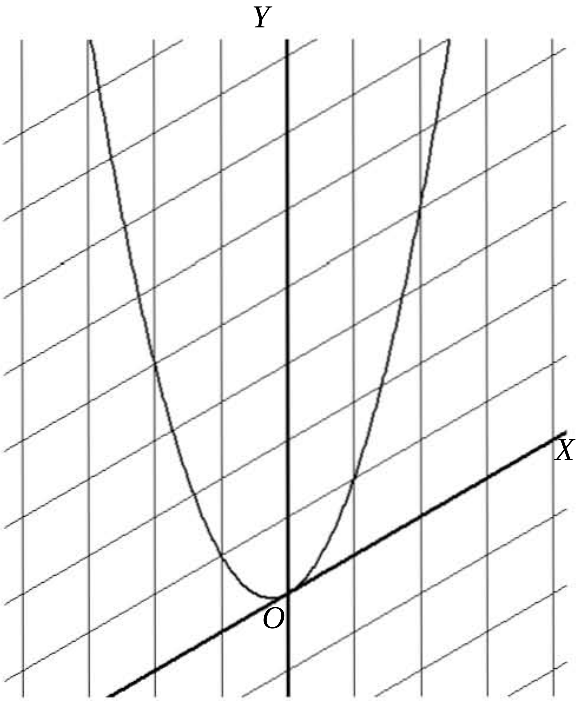
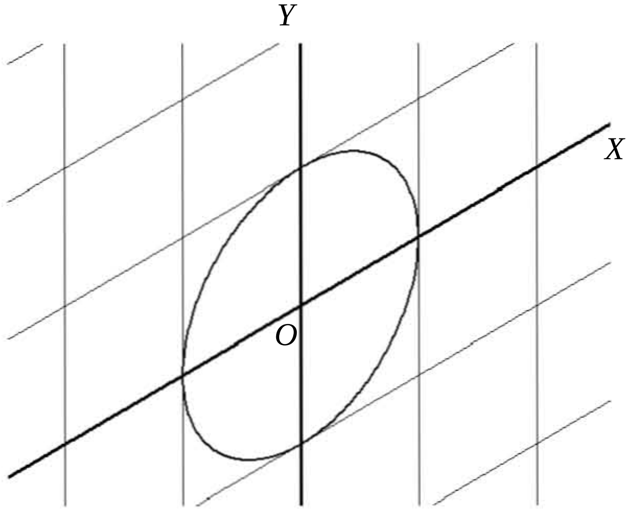


Beginning with $Y = X^2$, we obtain $y = \frac{4x^2 + \sqrt{3}x}{3} = \frac{4}{3}(x + \frac{\sqrt{3}}{8})^2 - \frac{1}{16}$, so the transformed curve is still a parabola, but it is *not* symmetrical about the *Y*-axis, having a line of symmetry at $x = -\frac{\sqrt{3}}{8}$ or $X = -\frac{1}{4}$, so the minimum point is $(-\frac{\sqrt{3}}{8}, -\frac{1}{16})_{Cartesian}$ or $(-\frac{1}{4}, \frac{1}{16})_{Isometric}$. (In the isometric plane, we take 'minimum point' to mean the position at which there is a *horizontal* tangent, rather than a tangent parallel to the *X*-axis, which, of course, happens at the origin, as in the Cartesian representation.)

3. Circles



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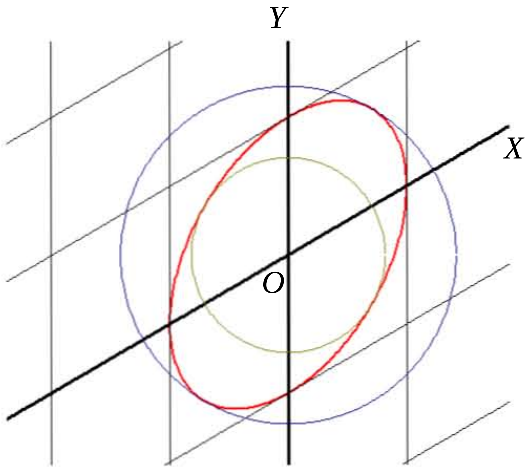

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