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96.08 Orthogonal mappings

The ordered pair (x, y) is commonly represented by a point in the Cartesian plane, but there are other possibilities. Let us instead represent (x, y) by the (directed, if we wish) line segment joining $(x, 0)$ to $(0, y)$ (Figure 1).

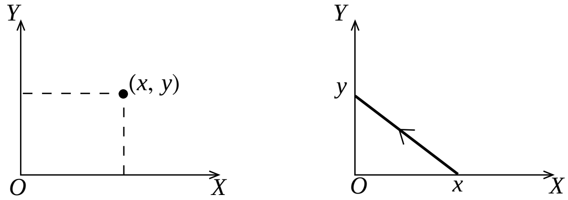
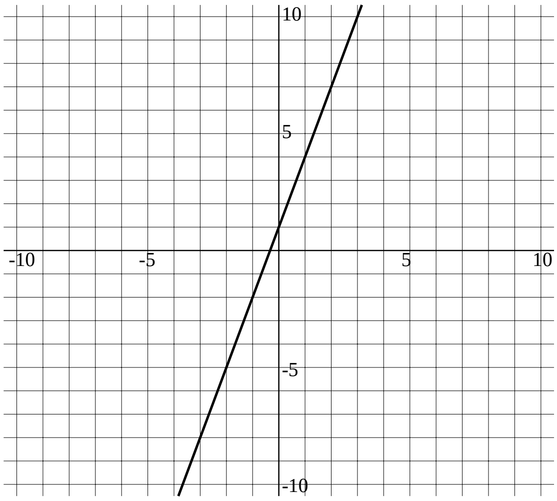
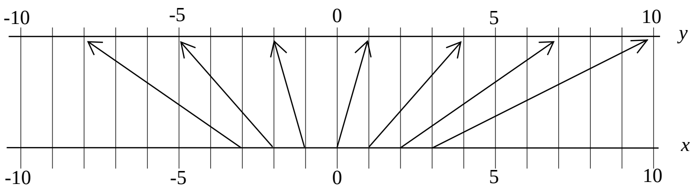


FIGURE 1: Alternative ways of representing the ordered pair (x, y): (a) as a point in the Cartesian plane; (b) as a (directed) line segment

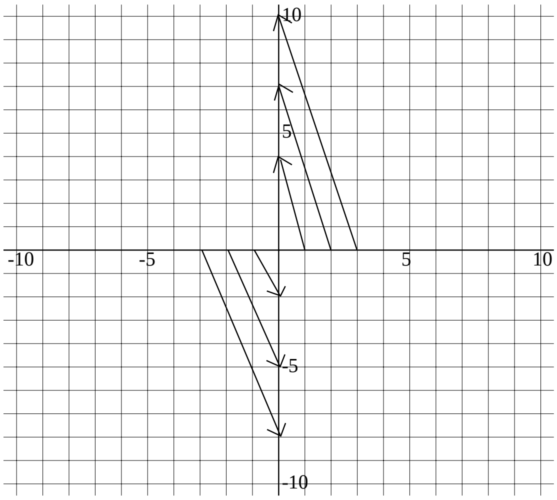
We can call this an *orthogonal mapping*, since the x-axis is being mapped onto a perpendicular y-axis. Mappings between parallel lines are well known, and the three approaches are represented in Figure 2 for the function $y = 3x + 1$.



$y = 3x + 1$ as a Cartesian graph



$y = 3x + 1$ as a mapping between parallel axes



$y = 3x + 1$ as an orthogonal mapping

FIGURE 2: Three ways of representing $y = 3x + 1$

What will functions look like when represented by orthogonal mappings? We will consider straight-line graphs $y = mx + c$, where m and c are constants. Instead of points $(t, mt + c)$, we will have line segments from $(t, 0)$ to $(0, mt + c)$. These line segments will have equations $\frac{y - 0}{x - t} = \frac{(mt + c) - 0}{0 - t}$, or $\left(m + \frac{c}{t}\right)x + y - (mt + c) = 0$, provided that $t \neq 0$. We can see that when $t = 0$, y will be undefined (corresponding to a vertical line segment), and as $x \to \pm\infty$, $y \to -mx + mt + c$, parallel lines of gradient $-m$ and intercept $(mt + c)$. If $m = 0$, $y = -\frac{c}{t}x + c$, a series of lines all passing through $(0, c)$. If $c = 0$, $y = -mx + mt$, which describes a series of parallel lines of gradient $-m$ and intercept mt. Varying the parameter t thus leads to a family of lines which envelope a curve. The situation with $m = -1$ (e.g., $x + y = 5$) is a well-known instance of curve stitching or string art (see Figure 3) [1].

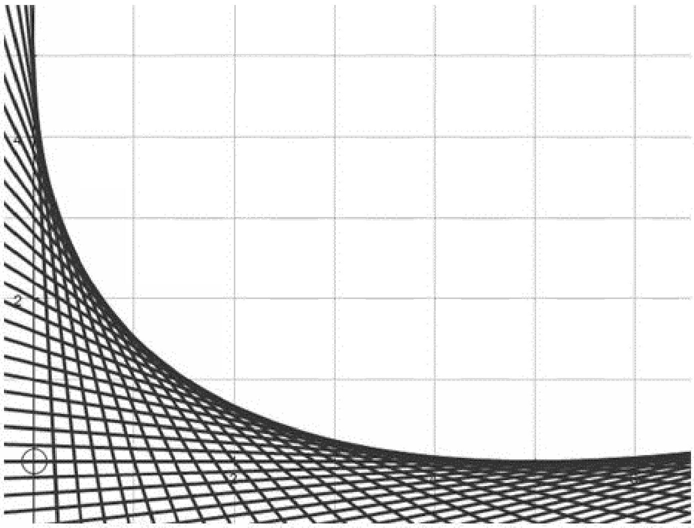


FIGURE 3: Curve stitching $x + y = 5$ leads to the envelope of a parabola

To find the envelope of a variable straight line given by $f(t)x + g(t)y + h(t) = 0$, where f, g and h are expressed in terms of a parameter t, we form the second equation $f'(t)x + g'(t)y + h'(t) = 0$ and eliminate t between these two equations. So, in our case, differentiating with respect to t gives $-\frac{c}{t^2}x - m = 0$, so $t^2 = -\frac{cx}{m}$. Rewriting the first equation gives $t(mx + y - c) + cx - mt^2 = 0$ and, since $mt^2 = -cx$, this is $t(mx + y - c) = -2cx$, so $t^2(mx + y - c)^2 = 4c^2x^2$, and substituting $t^2 = -\frac{cx}{m}$ gives $(mx + y - c)^2 + 4cmx = 0$ as the equation of the envelope.

This is a parabola, since it is of the form $Y^2 - kX = 0$, where k is a constant, with $Y = mx + y + \lambda$ and $X = x - my + \mu$. Comparing, we have $(mx + y - c)^2 + 4cmx = (mx + y + \lambda)^2 - k(x - my + \mu)$, and equating coefficients gives the values

$$k = \frac{-4cm}{1 + m^2}; \qquad \lambda = \frac{c(m^2 - 1)}{1 + m^2} \qquad \text{and} \qquad \mu = \frac{cm}{1 + m^2}.$$

This means that the axis of the parabola, $Y = 0$, is the line $mx + y + \lambda = 0$, i.e., $y = -mx + \frac{c(1 - m^2)}{1 + m^2}$, and the tangent at the vertex, $X = 0$, is the line $x - my + \mu = 0$, i.e., $y = \frac{x}{m} + \frac{c}{1 + m^2}$. Solving these equations simultaneously gives the coordinates of the vertex as $\left(\frac{-cm^3}{(1 + m^2)^2}, \frac{c}{(1 + m^2)^2}\right)$.

Acknowledgement

I would like to thank the referee for suggesting a simplification of the argument.

Reference

1. B. Bolt, Curve stitching − a mistaken curve, *Teaching Mathematics and its Applications*, **1**, 3, (1982) pp. 101-102.

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