96.08 Orthogonal mappings

The ordered pair \((x, y)\) is commonly represented by a point in the Cartesian plane, but there are other possibilities. Let us instead represent \((x, y)\) by the (directed, if we wish) line segment joining \((x, 0)\) to \((0, y)\) (Figure 1).

We can call this an orthogonal mapping, since the \(x\)-axis is being mapped onto a perpendicular \(y\)-axis. Mappings between parallel lines are well known, and the three approaches are represented in Figure 2 for the function \(y = 3x + 1\).
as a Cartesian graph

as a mapping between parallel axes

as an orthogonal mapping

FIGURE 2: Three ways of representing $y = 3x + 1$
What will functions look like when represented by orthogonal mappings? We will consider straight-line graphs $y = mx + c$, where $m$ and $c$ are constants. Instead of points $(t,0)$, we will have line segments from $(t,0)$ to $(0,mt+c)$. These line segments will have equations

$$\frac{y-0}{x-t} = \frac{(mt+c)-0}{0-t},$$

or

$$\left(m + \frac{c}{t}\right)x + y - (mt + c) = 0,$$

provided that $t \neq 0$. We can see that when $t = 0$, $y$ will be undefined (corresponding to a vertical line segment), and as $x \to \pm\infty$, $y \to -mx + mt + c$, parallel lines of gradient $-m$ and intercept $(mt + c)$. If $m = 0$, $y = -\frac{c}{t}x + c$, a series of lines all passing through $(0, c)$. If $c = 0$, $y = -mx + mt$, which describes a series of parallel lines of gradient $-m$ and intercept $mt$. Varying the parameter $t$ thus leads to a family of lines which envelope a curve. The situation with $m = -1$ (e.g., $x + y = 5$) is a well-known instance of curve stitching or string art (see Figure 3) [1].

**Figure 3:** Curve stitching $x + y = 5$ leads to the envelope of a parabola

To find the envelope of a variable straight line given by

$$f(t)x + g(t)y + h(t) = 0,$$

where $f$, $g$ and $h$ are expressed in terms of a parameter $t$, we form the second equation $f'(t)x + g'(t)y + h'(t) = 0$ and eliminate $t$ between these two equations. So, in our case, differentiating with respect to $t$ gives

$$\frac{c}{t^2} - m = 0,$$

so

$$t^2 = \frac{cx}{m}.$$

Rewriting the first equation gives

$$t(mx + y - c) + cx - m^2 = 0.$$

And, since

$$mt^2 = -cx,$$

this is

$$t(mx + y - c) = -2cx,$$

so

$$t^2(mx + y - c)^2 = 4c^2x^2,$$

and substituting

$$t^2 = \frac{cx}{m}$$

gives

$$(mx + y - c)^2 + 4cmx = 0$$

as the equation of the envelope.
This is a parabola, since it is of the form \( y^2 - kX = 0 \), where \( k \) is a constant, with \( Y = mx + y + \lambda \) and \( X = x - my + \mu \). Comparing, we have \( (mx + y - c)^2 + 4cmx = (mx + y + \lambda)^2 - k(x - my + \mu) \), and equating coefficients gives the values

\[
k = \frac{-4cm}{1 + m^2}; \quad \lambda = \frac{c(m^2 - 1)}{1 + m^2} \quad \text{and} \quad \mu = \frac{cm}{1 + m^2}.
\]

This means that the axis of the parabola, \( Y = 0 \), is the line \( mx + y + \lambda = 0 \), i.e., \( y = -mx + \frac{c(1 - m^2)}{1 + m^2} \), and the tangent at the vertex, \( X = 0 \), is the line \( x - my + \mu = 0 \), i.e., \( y = \frac{x}{m} + \frac{c}{1 + m^2} \). Solving these equations simultaneously gives the coordinates of the vertex as \( \left( \frac{-cm^3}{(1 + m^2)^2}, \frac{c}{(1 + m^2)^2} \right) \).

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Reference


COLIN FOSTER

*King Henry VIII School, Warwick Road, Coventry CV3 6AQ*

e-mail: c@foster77.co.uk