An offprint from
The Mathematical Gazette

Volume 96 Number 536 July 2012

96.30 Quadratic doublets

I have known for some time that $x^2 + 17x + 30$ is a convenient example when teaching quadratics, because it is a monic quadratic (one in which the coefficient of x^2 is 1) that factorises but can quickly be turned into a *non*-monic quadratic that also factorises, simply by changing the coefficient of x^2 from 1 to 2:

$$x^2 + 17x + 30 = (x + 2)(x + 15);$$

$$2x^2 + 17x + 30 = (2x + 5)(x + 6).$$

This works because 17 can be partitioned into 2 + 15 or into 5 + 12. A simpler example of the same thing is $x^2 + 7x + 6$, which relies on the fact that $7 = 1 + 6$ or $3 + 4$:

$$x^2 + 7x + 6 = (x + 1)(x + 6);$$

$$2x^2 + 7x + 6 = (2x + 3)(x + 2).$$

This set me wondering which other monic quadratics with integer coefficients would do this. It is important that the coefficient of x and the constant term are not both even, otherwise the non-monic version will merely reduce to a multiple of another monic quadratic (e.g., changing $x^2 + 14x + 24$ into $2x^2 + 14x + 24$ simply leads to a monic 'in disguise').

Considering the general case, we can begin with

$$(2x + a)(x + b) = 2x^2 + (a + 2b)x + ab$$

where a and b are non-zero integers. Now we change the coefficient of x^2 from 2 to 1 and we want

$$x^2 + (a + 2b)x + ab$$

to factorise. This will happen if, and only if, the discriminant is a perfect square; that is, if

$$(a + 2b)^2 - 4ab = k^2,$$

where k is an integer.

This gives

$$a^2 + (2b)^2 = k^2,$$

so a and $2b$ must form the two legs of a Pythagorean triangle. Since all primitive Pythagorean triples can be expressed as $(m^2 - n^2, 2mn, m^2 + n^2)$, where m and n are integers, at least one leg (the $2mn$ one) of any Pythagorean triangle is always even, so b can be assumed to be an integer.

Looking at the first few Pythagorean triples, and bearing in mind that a and b can be positive or negative, we obtain the quadratics listed in Table 1. (Shading indicates situations in which the non-monic quadratic is simply a multiple of a monic.) I find the pair $x^2 + 3x - 54$, $2x^2 + 3x - 54$ particularly beautiful because of the manner in which the 6 and 9 interplay.

Triple	a	b	Quadratic	a	b	Quadratic
(3, 4, 5)	3	2	$x^2 + 7x + 6 = (x+1)(x+6)$ $2x^2 + 7x + 6 = (2x+3)(x+2)$	−3	−2	$x^2 - 7x + 6 = (x-1)(x-6)$ $2x^2 - 7x + 6 = (2x-3)(x-2)$
	−3	2	$x^2 + x - 6 = (x+3)(x-2)$ $2x^2 + x - 6 = (2x-3)(x+2)$	3	−2	$x^2 - x - 6 = (x-3)(x+2)$ $2x^2 - x - 6 = (2x+3)(x-2)$
(6, 8, 10)	6	4	$x^2 + 14x + 24 = (x+2)(x+12)$ $2x^2 + 14x + 24 = 2(x^2 + 7x + 12)$ $= 2(x+3)(x+4)$	−6	−4	$x^2 - 14x + 24 = (x-2)(x-12)$ $2x^2 - 14x + 24 = 2(x^2 - 7x - 12)$ $= 2(x-3)(x-4)$
	−6	4	$x^2 + 2x - 24 = (x+6)(x-4)$ $2x^2 + 2x - 24 = 2(x^2 + x - 12)$ $= 2(x-3)(x+4)$	6	−4	$x^2 - 2x - 24 = (x-6)(x+4)$ $2x^2 - 2x - 24 = 2(x^2 - x - 12)$ $= 2(x+3)(x-4)$
(9, 12, 15)	9	6	$x^2 + 21x + 54 = (x+3)(x+18)$ $2x^2 + 21x + 54 = (2x+9)(x+6)$	−9	−6	$x^2 - 21x + 54 = (x-3)(x-18)$ $2x^2 - 21x + 54 = (2x-9)(x-6)$
	−9	6	$x^2 + 3x - 54 = (x+9)(x-6)$ $2x^2 + 3x - 54 = (2x-9)(x+6)$	9	−6	$x^2 + 3x - 54 = (x+9)(x-6)$ $2x^2 + 3x - 54 = (2x+9)(x-6)$
(5, 12, 13)	5	6	$x^2 + 17x + 30 = (x+2)(x+15)$ $2x^2 + 17x + 30 = (2x+5)(x+6)$	−5	−6	$x^2 - 17x + 30 = (x-2)(x-15)$ $2x^2 - 17x + 30 = (2x-5)(x-6)$
	−5	6	$x^2 + 7x - 30 = (x-3)(x+10)$ $2x^2 + 7x - 30 = (2x-5)(x+6)$	5	−6	$x^2 - 7x - 30 = (x+3)(x-10)$ $2x^2 - 7x - 30 = (2x+5)(x-6)$
(8, 15, 17)	15	4	$x^2 + 23x + 60 = (x+3)(x+20)$ $2x^2 + 23x + 60 = (2x+15)(x+4)$	−15	−4	$x^2 - 23x + 60 = (x-3)(x-20)$ $2x^2 - 23x + 60 = (2x-15)(x-4)$
	−15	4	$x^2 - 7x - 60 = (x-5)(x+12)$ $2x^2 - 7x - 60 = (2x-15)(x+4)$	15	−4	$x^2 + 7x - 60 = (x+5)(x-12)$ $2x^2 + 7x - 60 = (2x+15)(x-4)$

TABLE 1

It is interesting to consider whether this can be generalised to other coefficients of x^2. If the coefficient of x^2 is a prime p, then we can write

$$(px + a)(x + b) = px^2 + (a + pb)x + ab$$

and we want

$$x^2 + (a + pb)x + ab$$

to factorise, which will happen if, and only if,

$$(a + pb)^2 - 4ab = k^2,$$

where k is an integer. We can write this as

$$(a + b(p - 2))^2 + (2b)^2(p - 1) = k^2,$$

which is of Pythagorean triple form if, and only if, $p - 1$ is a perfect square. This means that we can construct quadratic doublets using Pythagorean triples only in cases where the coefficient of x^2 is 1 more than a square. This clearly includes our initial case, where $p = 2$, and the next case will be $p = 5$. Using the (5, 12, 13) Pythagorean triple leads to the doublet:

$$x^2 + 11x - 12 = (x + 12)(x - 1)$$
$$5x^2 + 11x - 12 = (5x - 4)(x + 3).$$

However, it is not immediately apparent that other cases, such as $p = 3$, are impossible. And when p is not prime, $(px + a)(x + b)$ is no longer the only possible factorisation. Nor is it clear whether it might be possible to find a quadratic *triplet* (or higher), in which the coefficient of x^2 can take *three* (or more) distinct integer values.

Acknowledgement

I am grateful to the editor and referee for directing my attention to [1], in which the situation where the related quadratics $x^2 + bx + c$ and $x^2 + bx - c$ both factorise is explored. Interestingly, the initial example given here, $2x^2 + 17x + 30$, also satisfies that condition, since $2x^2 + 17x - 30 = (2x - 3)(x + 10)$.

Reference

1. M. Harvey, $x^2 - 17x - 60 = 0$, *Math. Gaz.*, **81** (July 1997), pp. 267-269.

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