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96.47 Squares within squares

When a square is drawn with all of its vertices on the lattice points of a square grid, how many whole squares can be contained inside it? For example, for the tilted square shown in Figure 1, there are 13 whole squares inside. Is it possible to draw a square containing any given number of whole squares?

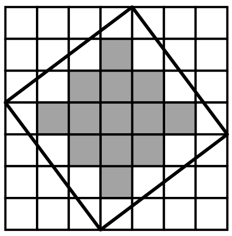

FIGURE 1: Thirteen whole squares inside a tilted square

Consider the four right-angled triangles around the edge, whose proportions define the slope of the tilting square, and let the lengths of their legs be a and b. Then Pythagoras' theorem gives us an upper bound of $a^2 + b^2$ for the maximum number of whole squares which can be inside the large square. Clearly this number will be achieved only when either a or b is zero and the sides of the square are parallel to the grid lines. For a *tilted* square, such as the one shown in Figure 1, we will obtain fewer whole squares, and for the case shown where $a = 3$ and $b = 4$ we can see the 13 visually as $2^2 + 3^2$ (Figure 2), suggesting that under certain circumstances the number of whole squares inside the tilted square will be

$$(a - 1)^2 + (b - 1)^2 = a^2 + b^2 - 2(a + b) + 2.$$

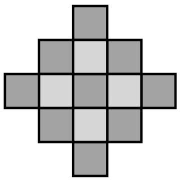

FIGURE 2: $13 = 2^2 + 3^2$

However, in general two consecutive squares need not be apparent, so we begin instead with the result that the number of squares cut by the diagonal of an $a \times b$ rectangle is given by $a + b - h$, where h is the highest common factor of a and b [1]. This means that the number *not* cut by the diagonal (the shaded ones in Figure 3) is $ab - (a + b) + h$.

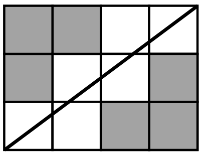

FIGURE 3: The squares in a rectangle that are *not* cut by the diagonal

So, by symmetry, the shaded squares *underneath* the diagonal line in Figure 3 will be $\frac{1}{2}(ab - (a + b) + h)$. This will always be an integer because the expression $ab - (a + b) + h$ will always be even, as shown in Table 1 (note that because the expression above is symmetrical in a and b it is not necessary to consider separately the case where a is even and b is odd):

a	b	ab	$a + b$	h
odd	odd	odd	even	odd
odd	even	even	odd	odd
even	even	even	even	even

TABLE 1: odd − even + odd = even − odd + odd = even − even + even = even

It follows from this that the number of whole squares *outside* the tilted square (Figure 4) will be $4 \times \frac{1}{2}(ab - (a + b) + h)$, so since the large square contains $(a + b)^2$ little squares, and the perimeter of the tilted square passes through $4(a + b - h)$ squares in total, the number of whole squares left *inside* the tilted square (Figure 1) is given in general by

$$\begin{aligned} & (a + b)^2 - 2(ab - (a + b) + h) - 4(a + b - h) \\ &= a^2 + b^2 - 2(a + b - h) \\ &= (a - 1)^2 + (b - 1)^2 + 2(h - 1). \end{aligned}$$

Thus, when a and b are coprime ($h = 1$), the number of whole squares inside the tilted square is a sum of two squares, as in the example discussed earlier where $a = 3$ and $b = 4$.

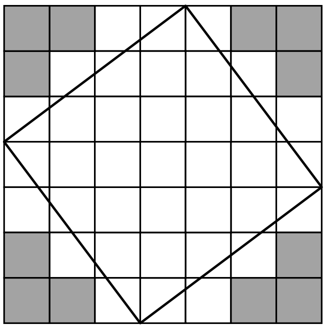

FIGURE 4: Shaded squares *outside* the tilted square

The number of whole squares for some different values of a and b are given in Table 2. The numbers of whole squares that can be contained inside a square drawn with all of its vertices on lattice points are: 0, 1, 4, 5, 9, 12, 13, 16, 17, 20, 24, 25, 28, 33, 36, 37, 40, 41, 45, 49, 52, 53, 60, 61, 64, 65, 72, 73, 76, 80, 81, 84, 85, 92, 93, 100, … [2]

$a\backslash b$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	0	1	4	9	16	25	36	49	64	81	100	121	144	169	196
2	1	4	5	12	17	28	37	52	65	84	101	124	145	172	197
3	4	5	12	13	20	33	40	53	72	85	104	129	148	173	204
4	9	12	13	24	25	36	45	64	73	92	109	136	153	180	205
5	16	17	20	25	40	41	52	65	80	105	116	137	160	185	220
6	25	28	33	36	41	60	61	76	93	108	125	156	169	196	225
7	36	37	40	45	52	61	84	85	100	117	136	157	180	217	232
8	49	52	53	64	65	76	85	112	113	132	149	176	193	220	245
9	64	65	72	73	80	93	100	113	144	145	164	189	208	233	264
10	81	84	85	92	105	108	117	132	145	180	181	204	225	252	285
11	100	101	104	109	116	125	136	149	164	181	220	221	244	269	296
12	121	124	129	136	137	156	157	176	189	204	221	264	265	292	321
13	144	145	148	153	160	169	180	193	208	225	244	265	312	313	340
14	169	172	173	180	185	196	217	220	233	252	269	292	313	364	365
15	196	197	204	205	220	225	232	245	264	285	296	321	340	365	420

TABLE 2: The number of whole squares contained in a square drawn with all its vertices on the lattice points of a square grid (a and b are defined in the text)

References

1. J. R. Branfield, An Investigation, *Math. Gaz*. **53** (October 1969), pp. 240-247
2. Sequence A194154, *The on-line encyclopedia of integer sequences*, http://oeis.org.

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