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96.07 Symmetrical cubics

It is common knowledge to sixth-form students that when a parabola crosses the x-axis the turning point is midway between the two x-intercepts. So if the roots are known, there is no need to complete the square or differentiate to find the coordinates of the turning point. Since, to a good approximation, parabolas describe the path of a projectile close to the surface of the earth, this means that over horizontal ground the range is twice the horizontal distance to the highest point, and this is an often-used short cut to solving mechanics problems.

Following some work exploiting this, a student was staring at a sketch of the graph of $y = x^3 - x$ (Figure 1) and wondering about the stationary points. "It doesn't work for cubics, does it?"

I suggested he use calculus to find the positions of the turning points, which he did, obtaining $x = \pm\frac{\sqrt{3}}{3}$, with magnitude clearly greater than $\frac{1}{2}$, confirming his visual impression that the turning points were nearer to the roots at ±1 than to the one at the origin.

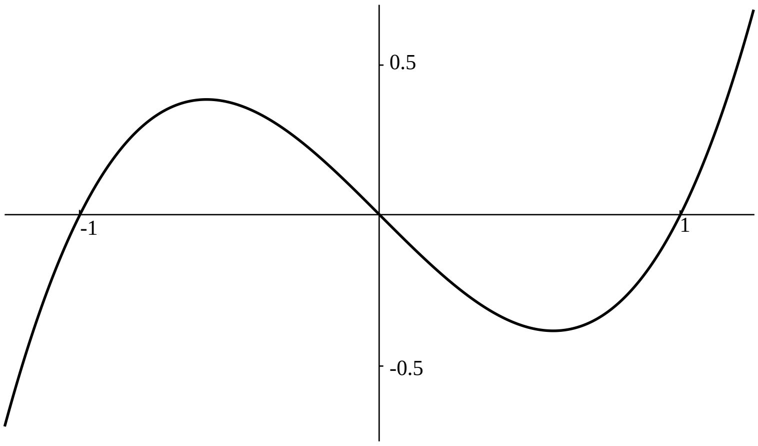

FIGURE 1: $y = x^3 - x$

He wondered whether translating the curve vertically might lead to two of the roots becoming equidistant from one of the turning points, but the shape did not lead him to think that this would work. So he wondered whether he could find a cubic in which a turning point *was* midway between the nearest two roots.

For the question to arise, the cubic must have two (and thus three) real roots, so we can consider the cubic

$$y = (x - a)(x - b)(px + q)$$

and differentiate, giving

$$\frac{dy}{dx} = (x - a)(x - b)p + \big(2x - (a + b)\big)(px + q).$$

So when $x = \frac{1}{2}(a + b)$, we have

$$\frac{dy}{dx} = -\frac{(a - b)^2 p}{4},$$

so $\frac{dy}{dx} = 0$ if, and only if, $a = b$, the situation of repeated roots where the curve touches the x-axis. So, other than in this trivial case, a cubic turning point cannot be midway between two roots.

A related question is to introduce a line $y = mx$, where m is a constant, such that the intersections of the line with the cubic occur equal x-distances either side of a turning point. For example, we can arrange that $y = x^3 - x$ and $y = mx$ do this if $x^3 - x = mx$ when $x = \frac{2}{3}\sqrt{3}$.

So $x\left(x^2 - (m + 1)\right) = 0$, giving $x = \pm\sqrt{m + 1}$, so $\sqrt{m + 1} = \frac{2}{3}\sqrt{3}$, giving $m = \frac{1}{3}$. This means that the required line is $y = \frac{1}{3}x$ (Figure 2).

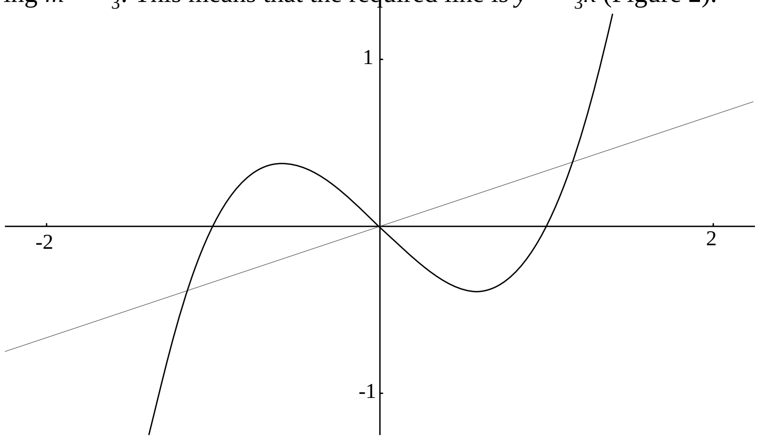

FIGURE 2: The curve $y = x^3 - x$ and the line $y = \frac{1}{3}x$

We failed in our original problem with cubics; however, for a quartic we can do it. We can consider a general quartic with roots a and b, having the equation

$$y = (x - a)(x - b)\left(px^2 + qx + r\right).$$

Differentiating this time gives

$$\frac{dy}{dx} = (x - a)(x - b)(2px + q) + \left(2x - (a + b)\right)\left(px^2 + qx + r\right).$$

So when $x = \frac{1}{2}(a + b)$, we have $\frac{dy}{dx} = -\frac{1}{4}(a - b)^2\left(p(a + b) + q\right)$, so $\frac{dy}{dx} = 0$ if, and only if, $q = -p(a + b)$.

A particular example would be where $a = 0$, $b = 2$, $p = 1$, $q = -2$ and $r = -1$, giving $y = x(x - 2)\left(x^2 - 2x - 1\right)$, as shown in Figure 3.

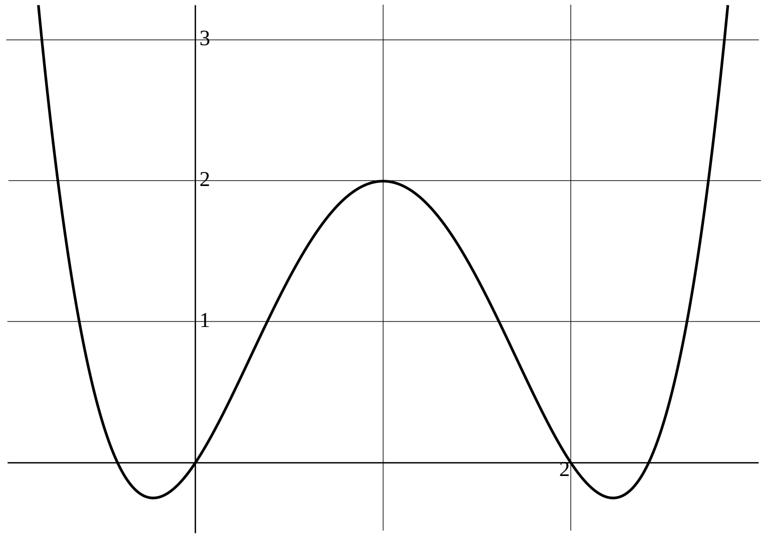

FIGURE 3: $y = x^4 - 4x^3 + 3x^2 + 2x$

Note that the remaining real roots (if any) are therefore given by

$$px^2 - (a + b)px + r = 0$$

and, since the sum of these roots is $(a + b)$, they are equidistant from $x = \frac{1}{2}(a + b)$, showing that the quartic is symmetrical about this line.

In fact, for quartics (and other even-degree polynomials), like quadratics, it is easy to find examples – the equation $y = x^4 - 1$, for instance. The real challenge was to do so for *odd*-degree polynomials. We had proved that it was impossible for cubics: would it be also impossible for a quintic, say? For higher-degree polynomials with two roots a and b, we can take $y = (x - a)(x - b)\left(p_n x^n + p_{n-1}x^{n-1} + \dots + p_0\right)$; i.e., $y = (x - a)(y - b)\sum_{i=0}^{n} p_i x^i$.

Now, $$\frac{dy}{dx} = (x - a)(y - b)\sum_{i=1}^{n} ip_i x^{i-1} + \left(2x - (a + b)\right)\sum_{i=0}^{n} p_i x^i.$$

Again, when $x = \frac{1}{2}(a + b)$, $\frac{dy}{dx} = -\frac{(a - b)^2}{4}\sum_{i=1}^{n} ip_i\left(\frac{a + b}{2}\right)^{i-1}$, and this can equal zero. For the quintic, $n = 3$, and choosing values $a = 0$ and $b = 2$, as before, gives the equation $3p_3 + 2p_2 + p_1 = 0$, so a possible solution would be $p_3 = 1$, $p_2 = -1$, $p_1 = -1$ and $p_0 = 5$, giving $y = x(x - 2)\left(x^3 - x^2 - x + 5\right)$, as shown in Figure 4.

So turning points can lie midway between roots for all polynomials *except* cubics. What originally seemed like a peculiar property of quadratics, turns out to be achievable (though by no means common) for everything except cubics!

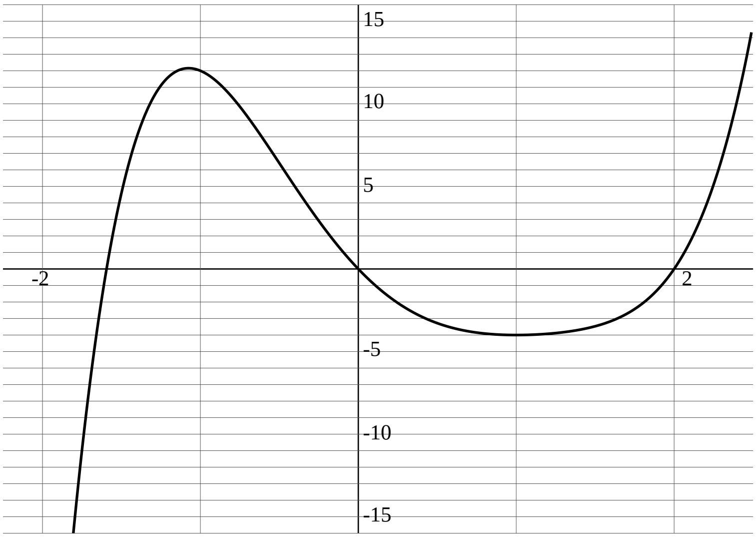

FIGURE 4: $y = x^5 - 3x^4 + x^3 + 7x^2 - 10x$

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COLIN FOSTER

King Henry VIII School, Warwick Road, Coventry, CV3 6AQ

e-mail: *c@foster77.co.uk*