An offprint from
The Mathematical Gazette
Volume 96 Number 536 March 2012

96.19 Taking a short cut through Pascal's triangle

Introduction

Everybody knows the proper way to travel through Pascal's triangle, starting at the top and moving down either left (L) or right (R), with each number in the triangle giving the total number of ways of reaching that position. But why restrict ourselves to left and right moves only, zigzagging our way down the triangle, when we could be going much more directly? We are crying out for *vertical* moves *downwards* to the next number below (that is, two rows lower). Let us call this sort of move D and consider the effect of having L, R *and* D on the number of possible ways of reaching each element. It somehow seems fitting that in a *tri*angle there should be *three* possible moves. Putting a 1 at the top and continuing down gives us what I will call the *short-cut triangle*, the first few rows of which are shown Figure 1.

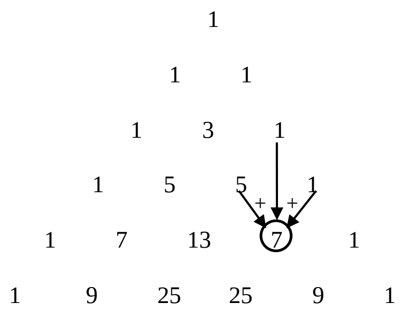


FIGURE 1: The first few rows of the short-cut triangle: $7 = 5 + 1 + 1$

The short-cut triangle

With the short-cut triangle, the number of ways ${}^nW_r$ of getting from the 1 at the top to the r th position in the n th row is the sum of three neighbours, rather than two:

$$ {}^nW_r = {}^{n-1}W_{r-1} + {}^{n-1}W_r + {}^{n-2}W_{r-1}. $$

We notice immediately that, in contrast to Pascal's triangle, all of the elements in the short-cut triangle are odd. This is inevitable since, away from the edges of the triangle, we are calculating each element by adding three odd numbers, which is necessarily odd. Near the edges, odd numbers are also obtained.

Analysing in more detail, one D move is equivalent to an RL or an LR, so a move to the rth position in the nth row, which in the conventional Pascal's triangle could happen via all the permutations of LLL...RRR (where there are r Ls and $(n - r)$ Rs), will now generally be possible in more ways. Supposing, without loss of generality, that $r \geqslant n - r$, we can make any positive integer i number of Ds we wish, up to and including $(n - r)$, which will leave, each time, $(r - i)$ Ls and $(r - n - i)$ Rs. From this it follows that ${}^{n-r}W_r = \sum_{i=0}^{r} \frac{(n-i)!}{i!(n-r-i)!(r-i)!}$, $r \leqslant \frac{n}{2}$. Since the rows of the short-cut triangle are necessarily symmetric, it follows that,

$$ {}^{n}W_r = {}^{n}W_{n-r} = \sum_{i=0}^{r} \frac{(n-i)!}{i!(n-r-i)!(r-i)!}, \ r \leqslant \tfrac{n}{2}. $$

For example,

$$ {}^{8}W_3 = \sum_{i=0}^{3} \frac{(8-i)!}{i!\,(3-i)!\,(5-i)!} $$

$$ = \frac{8!}{0!\,3!\,5!} + \frac{7!}{1!\,2!\,4!} + \frac{6!}{2!\,1!\,3!} + \frac{5!}{3!\,0!\,2!} = 231, $$

the four terms corresponding to the number of permutations of RRRLLLLL, DRRLLLL, DDRLLL and DDDLL respectively.

We note that the formula gives ${}^{n}W_0 = {}^{n}W_n = 1$, for all n, leading to the pattern of 1s on the edge diagonals. We can also establish results such as ${}^{n}W_1 = 2n - 1$, for all integer $n \geqslant 1$, which are the odd numbers seen along the first diagonal from the edge, and ${}^{n}W_2 = 2n^2 - 6n + 5 = (n-1)^2 + (n-2)^2$, for all integer $n \geqslant 2$, which are the centred square numbers, which appear along the second diagonal from the edge.

An alternative way of writing the formula would be ${}^{n}W_r = \sum_{i=0}^{r} {}^{r}C_i \, {}^{n-i}C_r$, which shows an interesting connection between the entries in the short-cut triangle and those in Pascal's triangle. For example, to calculate ${}^{8}W_3$ you just extract the appropriate elements from Pascal's triangle, form products and add, as shown in Figure 2.

Sums of rows

The sum of the elements in the nth row of Pascal's triangle is well known to be 2^n, since there are two possibilities (L and R) for each of the n steps. In the short-cut triangle, the totals of the rows are 1, 2, 5, 12, 29, 70, ..., which are the Pell numbers P_n, which satisfy the relation $P_n = 2P_{n-1} + P_{n-2}$, with $P_0 = 0$ and $P_1 = 0$. This arises because, for $n \geqslant 2$, each number in the nth row is the sum of the two numbers immediately above it in the $(n - 1)$th row, together with the number immediately above them in the $(n - 2)$th row. Since every number in the $(n - 1)$th row is counted exactly twice and every number in the $(n - 2)$th row is counted exactly once, summing all the numbers in the nth row gives

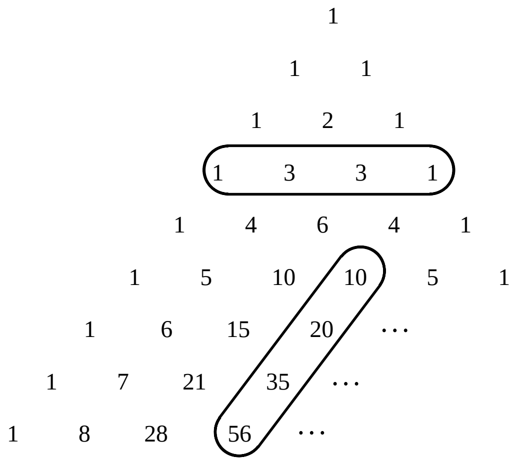
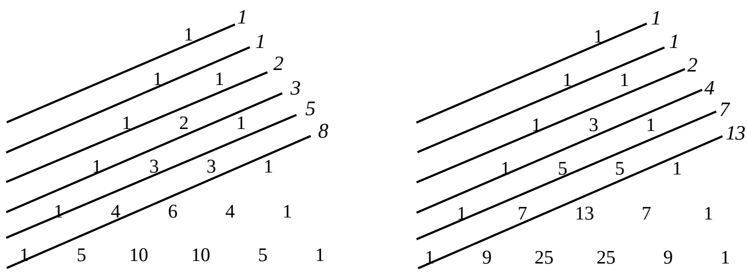


FIGURE 2: ${}^8W_3 = \sum_{i=0}^{3} {}^3C_i\,{}^{8-i}C_3 = \dot{1} \times 5\underset{\circ}{6} + \dot{3} \times 3\underset{\circ}{5} + \dot{3} \times 2\underset{\circ}{0} + \dot{1} \times 1\underset{\circ}{0} = 231$

twice the sum of the $(n - 1)$th row together with the sum of the $(n - 2)$th row, establishing the inductive definition for the Pell numbers, given that the zeroth row has a sum of 1 and the first row a sum of 2.

Sums of shallow diagonals

One of the more intriguing properties of Pascal's triangle is that the Fibonacci numbers appear as the sums of shallow diagonals. In the short-cut triangle, these sums instead give the Tribonacci numbers T_n, which satisfy the relation $T_n = T_{n-1} + T_{n-2} + T_{n-3}$, with $T_0 = 0$, $T_1 = 1$ and $T_2 = 1$ (see Figure 3).



(a) (b)

FIGURE 3: Sums of shallow diagonals giving
(a) Fibonacci numbers in Pascal's triangle,
and (b) Tribonacci numbers in the short-cut triangle

Fibonacci numbers arise in the case of Pascal's triangle because after the second shallow diagonal, each element of any shallow diagonal is the sum of an element in the previous shallow diagonal and an element in the shallow diagonal before that one. So the sum of any shallow diagonal (from the third onwards) is the sum of the two previous shallow diagonals, giving the inductive definition $F_n = F_{n-1} + F_{n-2}$ for Fibonacci numbers F_n, with $F_1 = 1$ and $F_2 = 1$, since these are the only numbers in the first two shallow diagonals.

In the short-cut triangle, a similar relationship means that, after the third shallow diagonal, each element of any shallow diagonal is the sum of one element from each of the three previous shallow diagonals, in such a way that no element is included more than once. This means that the Tribonacci relationship $T_n = T_{n-1} + T_{n-2} + T_{n-3}$, with $T_0 = 0$, $T_1 = 1$ and $T_2 = 1$, is satisfied and we obtain the Tribonacci sequence.

Acknowledgement

I would like to thank the anonymous referee for very helpful comments and suggestions.

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