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Odd and Even Fractions

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This article extends the familiar definitions of *odd* and *even* from the domain of the natural numbers to that of fractions (the rationals) and examines their behaviour under addition, subtraction, multiplication, and division.

Introduction

The terms *odd* and *even* are, of course, well-established for the natural numbers, and the behaviour of pairs of odd and even numbers under the four operations is elementary. But can we have odd and even *fractions*? Since, for fractions composed of integers, both the numerator and the denominator can be either even or odd, there are the following four possibilities (with nonzero denominator):

$$\frac{\text{even}}{\text{even}}, \quad \frac{\text{odd}}{\text{odd}}, \quad \frac{\text{even}}{\text{odd}}, \quad \frac{\text{odd}}{\text{even}}.$$

We immediately have a potential problem with equivalent fractions, since, for example,

$$\frac{4}{6} = \frac{\text{even}}{\text{even}}$$

but cancels down to

$$\frac{2}{3} = \frac{\text{even}}{\text{odd}}.$$

However, this difficulty arises only in connection with the $\frac{\text{even}}{\text{even}}$ case. Any fraction can become $\frac{\text{even}}{\text{even}}$ by multiplying the numerator and denominator by any even number, but the other three types are *stable* and cannot be interconverted. So a fraction of the type $\frac{\text{odd}}{\text{even}}$ can never be equal to another of the type $\frac{\text{odd}}{\text{odd}}$ or $\frac{\text{even}}{\text{odd}}$, and the same is true of the other two types. However, if we restrict our fractions to those in their simplest form, then we will exclude $\frac{\text{even}}{\text{even}}$, since 2, at least, must be a factor of both the numerator and the denominator. This leaves us with the following three separate distinct possibilities:

$$\frac{\text{odd}}{\text{odd}}, \quad \frac{\text{even}}{\text{odd}}, \quad \frac{\text{odd}}{\text{even}}.$$

Definitions

We can now choose to call $\frac{\text{odd}}{\text{odd}}$ an *odd fraction*, since in the case where the numerator is a multiple of the denominator, for example $\frac{15}{3}$, the result is an odd natural number. (Writing this natural number as a fraction with a denominator of 1 also replicates the $\frac{\text{odd}}{\text{odd}}$ structure.) For the same reason, it seems sensible to regard $\frac{\text{even}}{\text{odd}}$ as an *even fraction*, since, again when the numerator is a multiple of the denominator, for example $\frac{12}{3}$, an even number is the result. This just leaves $\frac{\text{odd}}{\text{even}}$, which can never give an integer answer. For this reason, I choose to regard this as neither odd nor even. For our purposes, even and odd are defined only if the denominator is odd, in which case the parity of the fraction is the same as that of the numerator. To summarise, we are making the following definitions:

$$\frac{\text{odd}}{\text{odd}} = \text{odd}, \qquad \frac{\text{even}}{\text{odd}} = \text{even}, \qquad \frac{\text{odd}}{\text{even}} = \text{neither}, \qquad \left(\frac{\text{even}}{\text{even}} = \text{unsimplified}\right).$$

Properties

With natural numbers under addition, table 1 applies. The same results obtain for subtraction, provided that we are comfortable with defining odd and even numbers on the negative integers. We now consider how odd and even *fractions* behave under addition and subtraction.

Adding two odd fractions leads to

$$\frac{\text{odd}}{\text{odd}} + \frac{\text{odd}}{\text{odd}} = \frac{\text{odd} \times \text{odd} + \text{odd} \times \text{odd}}{\text{odd} \times \text{odd}} = \frac{\text{odd} + \text{odd}}{\text{odd}} = \frac{\text{even}}{\text{odd}} = \text{even}.$$

With two even fractions, we have

$$\frac{\text{even}}{\text{odd}} + \frac{\text{even}}{\text{odd}} = \frac{\text{even} \times \text{odd} + \text{even} \times \text{odd}}{\text{odd} \times \text{odd}} = \frac{\text{even} + \text{even}}{\text{odd}} = \frac{\text{even}}{\text{odd}} = \text{even}.$$

With one of each, we have

$$\frac{\text{odd}}{\text{odd}} + \frac{\text{even}}{\text{odd}} = \frac{\text{odd} \times \text{odd} + \text{even} \times \text{odd}}{\text{odd} \times \text{odd}} = \frac{\text{odd} + \text{even}}{\text{odd}} = \frac{\text{odd}}{\text{odd}} = \text{odd}.$$

So we see behaviour identical to that displayed by the natural numbers, and subtraction will work similarly, with the same provisos as above.

Multiplication gives us

$$\begin{aligned}
\text{odd} \times \text{odd} &= \frac{\text{odd}}{\text{odd}} \times \frac{\text{odd}}{\text{odd}} = \frac{\text{odd}}{\text{odd}} = \text{odd},\\
\text{even} \times \text{odd} &= \frac{\text{even}}{\text{odd}} \times \frac{\text{odd}}{\text{odd}} = \frac{\text{even}}{\text{odd}} = \text{even},\\
\text{even} \times \text{even} &= \frac{\text{even}}{\text{odd}} \times \frac{\text{even}}{\text{odd}} = \frac{\text{even}}{\text{odd}} = \text{even},
\end{aligned}$$

again replicating the behaviour of the natural numbers.

Table 1

+	**odd**	**even**
odd	even	odd
even	odd	even

However, it would be nice to integrate the *neithers*, of the form $\frac{\text{odd}}{\text{even}}$, into the structure, if possible. We find that

$$\begin{aligned}
\text{neither} + \text{odd} &= \frac{\text{odd}}{\text{even}} + \frac{\text{odd}}{\text{odd}} \\
&= \frac{\text{odd} \times \text{odd} + \text{odd} \times \text{even}}{\text{even} \times \text{odd}} \\
&= \frac{\text{odd} + \text{even}}{\text{even}} \\
&= \frac{\text{odd}}{\text{even}} \\
&= \text{neither}, \\
\text{neither} + \text{even} &= \frac{\text{odd}}{\text{even}} + \frac{\text{even}}{\text{odd}} \\
&= \frac{\text{odd} \times \text{odd} + \text{even} \times \text{even}}{\text{even} \times \text{odd}} \\
&= \frac{\text{odd} + \text{even}}{\text{even}} \\
&= \frac{\text{odd}}{\text{even}} \\
&= \text{neither}.
\end{aligned}$$

However,

$$\begin{aligned}
\text{neither} + \text{neither} &= \frac{\text{odd}}{\text{even}} + \frac{\text{odd}}{\text{even}} \\
&= \frac{\text{odd} \times \text{even} + \text{odd} \times \text{even}}{\text{even} \times \text{even}} \\
&= \frac{\text{even} + \text{even}}{\text{even}} \\
&= \frac{\text{even}}{\text{even}} \\
&= ?
\end{aligned}$$

So with the addition of two neithers, we have an ambiguous case. An example would be

$$\begin{aligned}
\tfrac{3}{8} + \tfrac{1}{2} &= \tfrac{7}{8} \text{ (neither)}, \\
\tfrac{1}{6} + \tfrac{1}{2} &= \tfrac{2}{3} \text{ (even)}, \\
\tfrac{1}{10} + \tfrac{1}{2} &= \tfrac{3}{5} \text{ (odd)}.
\end{aligned}$$

Considering multiplication, we have

$$\text{neither} \times \text{neither} = \frac{\text{odd}}{\text{even}} \times \frac{\text{odd}}{\text{even}} = \frac{\text{odd}}{\text{even}} = \text{neither},$$

$$\text{neither} \times \text{odd} = \frac{\text{odd}}{\text{even}} \times \frac{\text{odd}}{\text{odd}} = \frac{\text{odd}}{\text{even}} = \text{neither}.$$

However, this time

$$\text{neither} \times \text{even} = \frac{\text{odd}}{\text{even}} \times \frac{\text{even}}{\text{odd}} = \frac{\text{even}}{\text{even}} = ?$$

So under multiplication it is with evens that the neithers lead to ambiguity. An example would be

$$\tfrac{3}{8} \times \tfrac{2}{5} = \tfrac{3}{20} \text{ (neither)},$$
$$\tfrac{1}{2} \times \tfrac{4}{5} = \tfrac{2}{5} \text{ (even)},$$
$$\tfrac{1}{2} \times \tfrac{2}{5} = \tfrac{1}{5} \text{ (odd)}.$$

So the neithers are *dominant*, in the sense that combination by addition with odd or even fractions, or by multiplication with odd fractions, or other neithers, leads to neithers. However, adding two neithers, or multiplying a neither by an even fraction, can lead to odd, even, or neither fractions.

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Extra-magic squares – part 2

5	31	35	60	57	34	8	30
19	9	53	46	47	56	18	12
16	22	42	39	52	61	27	1
63	37	25	24	3	14	44	50
26	4	64	49	38	43	13	23
41	51	15	2	21	28	62	40
54	48	20	11	10	17	55	45
36	58	6	29	32	7	33	59

This magic square does not only have all its row-sums and column-sums as well as the two diagonal sums equal, to 260, but the sums of the squares of the entries in each row, column, and diagonal are also equal, at 11 180.

Reference

1 Kazem Faeghi, *The Mathematical Amusements and Games.*

Students' Investigation House,
Shariati Avenue, Sirjan, Iran

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