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Trigonometry *Without* Right Angles

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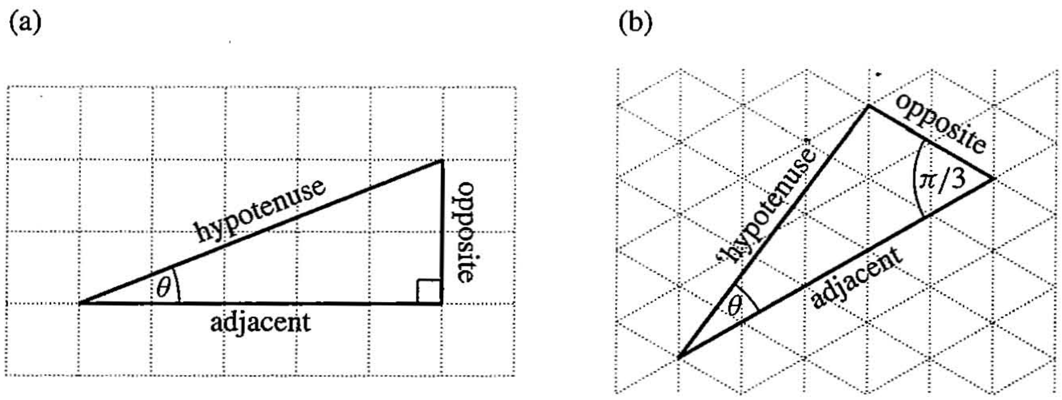
This article describes how trigonometric functions can be defined for non-right-angled triangles. In particular, the ratios esin, ecos, and etan, corresponding to the sine, cosine, and tangent ratios, are defined for triangles containing a $\pi/3$ angle. Some of the implications are discussed, including the graphs of these functions and the small-angle approximations.

How important are right angles for trigonometry? The standard trigonometric functions are defined as the ratios of sides within right-angled triangles, but is this essential? On a square lattice it makes sense to draw right-angled triangles and label the sides 'opposite', 'adjacent', and 'hypotenuse', but what about on an *isometric* grid?

We will label the sides of a triangle containing a $\frac{\pi}{3}$ angle as 'adjacent' and 'opposite' to a given angle (θ in figure 1) and call the side opposite the $\frac{\pi}{3}$ angle the 'hypotenuse'. Then we can define 'equilateral' trigonometric functions esin, ecos, and etan by analogy with the sin, cos, and tan functions.

We can express the e-trigonometric functions in terms of the ordinary trigonometric functions by using the sine rule and the cosine rule on the $\frac{\pi}{3}$ triangle. Using o for opposite, a for adjacent, and h for 'hypotenuse' in the $\frac{\pi}{3}$ triangle we have, from the sine rule, that

$$\frac{o}{\sin\theta} = \frac{h}{\sin(\frac{\pi}{3})} = \frac{a}{\sin(\pi - \frac{\pi}{3} - \theta)}. \tag{1}$$



$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \text{esin}\,\theta = \frac{\text{opposite}}{\text{'hypotenuse'}}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \text{ecos}\,\theta = \frac{\text{adjacent}}{\text{'hypotenuse'}}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} \qquad \text{etan}\,\theta = \frac{\text{opposite}}{\text{adjacent}}$$

Figure 1 (a) Trigonometry in a right-angled triangle; (b) e-trigonometry in a $\frac{\pi}{3}$ triangle.

The first equality in (1) reduces to

$$\frac{o}{\sin\theta} = \frac{2\sqrt{3}h}{3},$$

so

$$\operatorname{esin}\theta = \frac{o}{h} = \frac{2\sqrt{3}}{3}\sin\theta. \tag{2}$$

So the esine function is simply proportional to the ordinary sine function (and the constant of proportionality is not very different from 1).

The second inequality in (1) gives

$$\operatorname{ecos}\theta = \frac{a}{h} = \frac{\sin(\frac{2\pi}{3}-\theta)}{\sin(\frac{\pi}{3})} = \frac{\frac{\sqrt{3}}{2}\cos\theta + \frac{1}{2}\sin\theta}{\frac{\sqrt{3}}{2}} = \cos\theta + \frac{\sqrt{3}}{3}\sin\theta, \tag{3}$$

which we could also write as

$$\operatorname{ecos}\theta = \frac{2\sqrt{3}}{3}\sin\left(\theta + \frac{\pi}{3}\right).$$

This time there is the same factor of $\frac{2\sqrt{3}}{3}$ at the front, so the ecos function has the same amplitude as the esin function, but there is a phase difference of $\frac{\pi}{3}$. (This contrasts with the phase difference of $\frac{\pi}{2}$ between the sin and cos functions.)

Finally, from (1) we obtain

$$\begin{aligned}\operatorname{etan}\theta &= \frac{o}{a}\\ &= \frac{\sin\theta}{\sin(\frac{2\pi}{3}-\theta)}\\ &= \frac{\sin\theta}{\sin(\frac{2\pi}{3})\cos\theta - \cos(\frac{2\pi}{3})\sin\theta}\\ &= \frac{\sin\theta}{\frac{\sqrt{3}}{2}\cos\theta + \frac{1}{2}\sin\theta}\\ &= \frac{2}{1+\sqrt{3}\cot\theta}.\end{aligned} \tag{4}$$

The graphs of $y = \operatorname{esin}\theta$, $y = \operatorname{ecos}\theta$, and $y = \operatorname{etan}\theta$ are shown in figure 2.

Notice that the three graphs are coincident at $(\frac{\pi}{3} + 2n\pi, 1)$, where n is an integer, where $\operatorname{esin}\theta = \operatorname{ecos}\theta = \operatorname{etan}\theta$, corresponding to an equilateral triangle in which all three ratios are 1. This contrasts with the fact that there are no values of θ for which $\sin\theta = \cos\theta = \tan\theta$.

Now using the cosine rule, we have

$$h^2 = o^2 + a^2 - 2ao\cos(\tfrac{\pi}{3}),$$

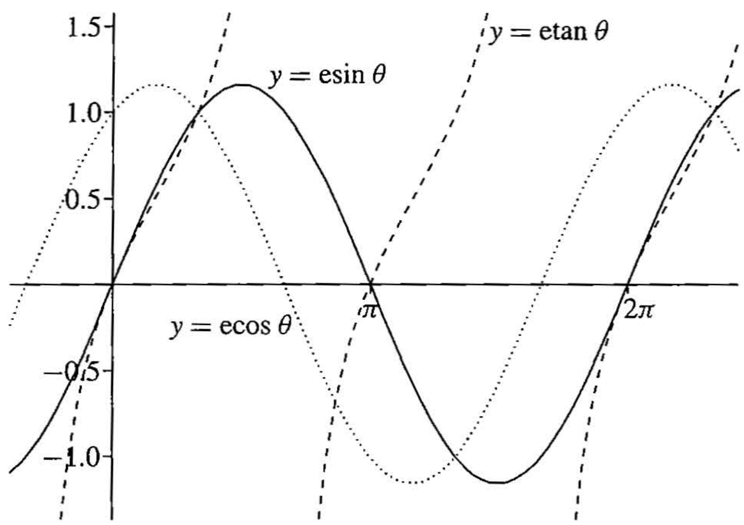


Figure 2 The graphs of $y = \mathrm{esin}\,\theta$, $y = \mathrm{ecos}\,\theta$, and $y = \mathrm{etan}\,\theta$.

which reduces to $h^2 = o^2 + a^2 - ao$. Dividing throughout by h^2 gives

$$\mathrm{esin}^2\,\theta + \mathrm{ecos}^2\,\theta - \mathrm{esin}\,\theta\,\mathrm{ecos}\,\theta = 1. \tag{5}$$

Alternatively, we could write

$$h^2 = o^2 + a^2 - ao = (o-a)^2 + ao.$$

Adding these equations, we obtain $2h^2 = o^2 + a^2 + (o-a)^2$ and dividing through by h^2 this time gives

$$\mathrm{esin}^2\,\theta + \mathrm{ecos}^2\,\theta + (\mathrm{esin}\,\theta - \mathrm{ecos}\,\theta)^2 = 2.$$

From (2), we can see that, for small values of θ in radians,

$$\mathrm{esin}\,\theta \approx \tfrac{2\sqrt{3}}{3}\theta.$$

From (5), using this approximation for $\mathrm{esin}\,\theta$, we have that

$$\mathrm{ecos}^2\,\theta - \tfrac{2\sqrt{3}}{3}\theta\,\mathrm{ecos}\,\theta + \left(\tfrac{2\sqrt{3}}{3}\theta\right)^2 - 1 = 0,$$

so

$$\mathrm{ecos}^2\,\theta - \tfrac{2\sqrt{3}}{3}\theta\,\mathrm{ecos}\,\theta + \tfrac{4}{3}\theta^2 - 1 = 0.$$

Completing the square, we have

$$\left(\mathrm{ecos}\,\theta - \tfrac{\sqrt{3}}{3}\theta\right)^2 - \tfrac{1}{3}\theta^2 + \tfrac{4}{3}\theta^2 - 1 = 0,$$

so

$$\left(\mathrm{ecos}\,\theta - \tfrac{\sqrt{3}}{3}\theta\right)^2 = 1 - \theta^2,$$

giving $\mathrm{ecos}\,\theta = \frac{\sqrt{3}}{3}\theta + \sqrt{1-\theta^2}$, taking the positive square root. Expanding the root binomially, we end up with

$$\mathrm{ecos}\,\theta \approx 1 + \tfrac{\sqrt{3}}{3}\theta - \tfrac{1}{2}\theta^2,$$

ignoring terms in θ^4 and above. As a check on our working, we can obtain the same result more straightforwardly from (3), since

$$\operatorname{ecos}\theta = \cos\theta + \tfrac{\sqrt{3}}{3}\sin\theta \approx \left(1 - \tfrac{1}{2}\theta^2\right) + \tfrac{\sqrt{3}}{3}\theta,$$

using the usual small-angle approximations for $\sin\theta$ and $\cos\theta$. Interestingly, the version with the binomial approximation to the square root gives a much better approximation for $\theta > \frac{\pi}{6}$ than the original approximation does,

$$\operatorname{ecos}\theta \approx \frac{\sqrt{3}}{3}\theta + \sqrt{1-\theta^2},$$

with

$$\operatorname{ecos}\theta \approx 1 + \frac{\sqrt{3}}{3}\theta - \tfrac{1}{2}\theta^2$$

not leading to huge discrepancies until θ is significantly greater than $\frac{2\pi}{3}$.

Finally, from (4) when θ is small we have

$$\operatorname{etan}\theta \approx \frac{2}{\sqrt{3}\cot\theta} = \frac{2\sqrt{3}}{3}\tan\theta,$$

so $\operatorname{etan}\theta \approx \frac{2\sqrt{3}}{3}\theta$. Since $\frac{2\sqrt{3}}{3}$ is fairly close to 1, in fact we are not very far from

$$\operatorname{esin}\theta \approx \operatorname{etan}\theta \approx \sin\theta \approx \tan\theta \approx \theta$$

anyway.

In a world without right angles we could still do e-trigonometry and solve triangles in a similar way to the familiar one. We could use a calculator with esin, ecos, and etan buttons (or e-trigonometry tables) just as easily.

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How good is your algebra?

$$\begin{aligned} p &= a(x^2-1) + 2b(x+1) - 2c(x-1),\\ q &= b(x^2-1) + 2c(x+1) - 2a(x-1),\\ r &= c(x^2-1) + 2a(x+1) - 2b(x-1). \end{aligned}$$

What is $p^2 + q^2 + r^2$?

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