This article describes how trigonometric functions can be defined for non-right-angled triangles. In particular, the ratios esin, ecos, and etan, corresponding to the sine, cosine, and tangent ratios, are defined for triangles containing a $\pi/3$ angle. Some of the implications are discussed, including the graphs of these functions and the small-angle approximations.

How important are right angles for trigonometry? The standard trigonometric functions are defined as the ratios of sides within right-angled triangles, but is this essential? On a square lattice it makes sense to draw right-angled triangles and label the sides ‘opposite’, ‘adjacent’, and ‘hypotenuse’, but what about on an isometric grid?

We will label the sides of a triangle containing a $\pi/3$ angle as ‘adjacent’ and ‘opposite’ to a given angle ($\theta$ in figure 1) and call the side opposite the $\pi/3$ angle the ‘hypotenuse’. Then we can define ‘equilateral’ trigonometric functions esin, ecos, and etan by analogy with the sin, cos, and tan functions.

We can express the e-trigonometric functions in terms of the ordinary trigonometric functions by using the sine rule and the cosine rule on the $\pi/3$ triangle. Using $o$ for opposite, $a$ for adjacent, and $h$ for ‘hypotenuse’ in the $\pi/3$ triangle we have, from the sine rule, that

$$\frac{o}{\sin \theta} = \frac{h}{\sin(\pi/3)} = \frac{a}{\sin(\pi - \pi/3 - \theta)}.$$  (1)

(a) Trigonometry in a right-angled triangle; (b) e-trigonometry in a $\pi/3$ triangle.
The first equality in (1) reduces to
\[ \frac{o}{\sin \theta} = \frac{2\sqrt{3}h}{3}, \]
so
\[ \text{esin } \theta = \frac{o}{h} = \frac{2\sqrt{3}}{3} \sin \theta. \]  
(2)

So the esine function is simply proportional to the ordinary sine function (and the constant of proportionality is not very different from 1).

The second inequality in (1) gives
\[ \text{ecos } \theta = \frac{a}{h} = \frac{\sin \left(\frac{2\pi}{3} - \theta\right)}{\sin \left(\frac{\pi}{3}\right)} = \frac{\frac{\sqrt{3}}{2} \cos \theta + \frac{1}{2} \sin \theta}{\frac{\sqrt{3}}{2}} = \cos \theta + \frac{\sqrt{3}}{3} \sin \theta, \]
which we could also write as
\[ \text{ecos } \theta = \frac{2\sqrt{3}}{3} \sin \left(\theta + \frac{\pi}{3}\right). \]

This time there is the same factor of \( \frac{2\sqrt{3}}{3} \) at the front, so the ecos function has the same amplitude as the esin function, but there is a phase difference of \( \frac{\pi}{3} \). (This contrasts with the phase difference of \( \frac{\pi}{2} \) between the sin and cos functions.)

Finally, from (1) we obtain
\[ \text{etan } \theta = \frac{o}{a} \]
\[ = \frac{\sin \theta}{\sin \left(\frac{2\pi}{3} - \theta\right)} \]
\[ = \frac{\sin \theta}{\sin \left(\frac{2\pi}{3}\right) \cos \theta - \cos \left(\frac{2\pi}{3}\right) \sin \theta} \]
\[ = \frac{\frac{\sqrt{3}}{2} \cos \theta + \frac{1}{2} \sin \theta}{\frac{\sqrt{3}}{2} \cos \theta + \frac{1}{2} \sin \theta} \]
\[ = \frac{2}{1 + \sqrt{3} \cot \theta}. \]  
(4)

The graphs of \( y = \text{esin } \theta, y = \text{ecos } \theta, \) and \( y = \text{etan } \theta \) are shown in figure 2.

Notice that the three graphs are coincident at \( \left(\frac{\pi}{3} + 2n\pi, 1\right) \), where \( n \) is an integer, where \( \text{esin } \theta = \text{ecos } \theta = \text{etan } \theta \), corresponding to an equilateral triangle in which all three ratios are 1. This contrasts with the fact that there are no values of \( \theta \) for which \( \sin \theta = \cos \theta = \tan \theta \).

Now using the cosine rule, we have
\[ h^2 = o^2 + a^2 - 2oa \cos \left(\frac{\pi}{3}\right), \]
which reduces to \( h^2 = \sigma^2 + a^2 - ao \). Dividing throughout by \( h^2 \) gives
\[
esin^2 \theta + \cos^2 \theta - \sin \theta \cos \theta = 1.
\] (5)

Alternatively, we could write
\[
h^2 = \sigma^2 + a^2 - ao = (\sigma - a)^2 + ao.
\]

Adding these equations, we obtain \( 2h^2 = \sigma^2 + a^2 + (\sigma - a)^2 \) and dividing through by \( h^2 \) this time gives
\[
esin^2 \theta + \cos^2 \theta + (\sin \theta - \cos \theta)^2 = 2.
\]

From (2), we can see that, for small values of \( \theta \) in radians,
\[
esin \theta \approx \frac{2\sqrt{3}}{3} \theta.
\]

From (5), using this approximation for \( \sin \theta \), we have that
\[
\cos^2 \theta - \frac{2\sqrt{3}}{3} \theta \cos \theta + \left(\frac{2\sqrt{3}}{3} \theta\right)^2 - 1 = 0,
\]
so
\[
\cos^2 \theta - \frac{2\sqrt{3}}{3} \theta \cos \theta + \frac{4}{3} \theta^2 - 1 = 0.
\]

Completing the square, we have
\[
(\cos \theta - \frac{\sqrt{3}}{3} \theta)^2 - \frac{1}{3} \theta^2 + \frac{4}{3} \theta^2 - 1 = 0,
\]
so
\[
(\cos \theta - \frac{\sqrt{3}}{3} \theta)^2 = 1 - \theta^2,
\]
giving \( \cos \theta = \frac{\sqrt{3}}{3} \theta + \sqrt{1 - \theta^2} \), taking the positive square root. Expanding the root binomially, we end up with
\[
\cos \theta \approx 1 + \frac{\sqrt{3}}{3} \theta - \frac{1}{2} \theta^2,
\]
ignoring terms in \( \theta^4 \) and above. As a check on our working, we can obtain the same result more straightforwardly from (3), since

\[
\text{ecos } \theta = \cos \theta + \frac{\sqrt{3}}{3} \sin \theta \approx \left(1 - \frac{1}{2} \theta^2\right) + \frac{\sqrt{3}}{3} \theta,
\]

using the usual small-angle approximations for \( \sin \theta \) and \( \cos \theta \). Interestingly, the version with the binomial approximation to the square root gives a much better approximation for \( \theta > \frac{\pi}{6} \) than the original approximation does,

\[
\text{ecos } \theta \approx \frac{\sqrt{3}}{3} \theta + \sqrt{1 - \theta^2},
\]

with

\[
\text{ecos } \theta \approx 1 + \frac{\sqrt{3}}{3} \theta - \frac{1}{2} \theta^2
\]

not leading to huge discrepancies until \( \theta \) is significantly greater than \( \frac{2\pi}{3} \).

Finally, from (4) when \( \theta \) is small we have

\[
\text{etan } \theta \approx \frac{2}{\sqrt{3} \cot \theta} = \frac{2\sqrt{3}}{3} \tan \theta,
\]

so \( \text{etan } \theta \approx \frac{2\sqrt{3}}{\theta} \). Since \( \frac{2\sqrt{3}}{\theta} \) is fairly close to 1, in fact we are not very far from

\[
\text{esin } \theta \approx \text{etan } \theta \approx \sin \theta \approx \tan \theta \approx \theta
\]

anyway.

In a world without right angles we could still do e-trigonometry and solve triangles in a similar way to the familiar one. We could use a calculator with esin, ecos, and etan buttons (or e-trigonometry tables) just as easily.

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**How good is your algebra?**

\[
p = a(x^2 - 1) + 2b(x + 1) - 2c(x - 1),
\]

\[
q = b(x^2 - 1) + 2c(x + 1) - 2a(x - 1),
\]

\[
r = c(x^2 - 1) + 2a(x + 1) - 2b(x - 1).
\]

What is \( p^2 + q^2 + r^2 \)?