A Different Pictorial Way of Looking at Simultaneous Equations

by Colin Foster

Simultaneous equations in two unknowns are normally represented graphically by a pair of straight lines. In each equation, x and y are treated as variables, and where the lines cross, both equations are satisfied simultaneously and the coordinates give the solution (see Foster, 2003). (For example, in Figure 1 we have the graphs of $x + y = 5$ and $x - y = 1$ and their solution $x = 3$, $y = 2$.)
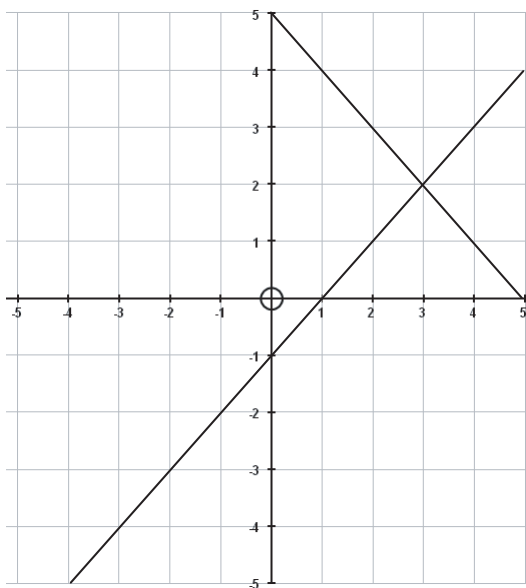

Fig. 1

The Structure of the Diagrams

I would like to offer an alternative representation which may be helpful in coming to terms with the concept of simultaneous equations. It is easier to understand the idea initially if you already know the solution and are trying to construct a pair of simultaneous equations of the form $ax + by = c$ which have this solution. [This can be an interesting task for pupils to work on, providing quite different opportunities from those presented by the more usual task of solving ready-made equations (see Watson and Mason, 2005).] To illustrate the idea, we will use the same solution as in the example in Figure 1; i.e. $x = 3$ and $y = 2$. Instead of plotting *possible* values of x and y for *fixed* values of a, b and c, we write down the value of c at each point (a, b), evaluating it for the *known* values of x and y (the solution).

For example, the equation $x + y = 5$ is represented by the value of 5 written at the grid point $(1, 1)$ and $4x - 5y = 2$ is represented by the value 2 written at the grid point $(4, -5)$. So as we move along the x-axis we are constructing equations 'with a greater number of xs in them' and similarly for ys along the y-axis. Each possible equation (with this solution) corresponds to a point in the plane. Filling in all the grid points for integer values of a and b from -5 to 5 (but still with $x = 3$ and $y = 2$ throughout) produces the diagram shown in Figure 2 (see Note 1). It is a good idea to take a moment to be sure that you see what this figure is showing!
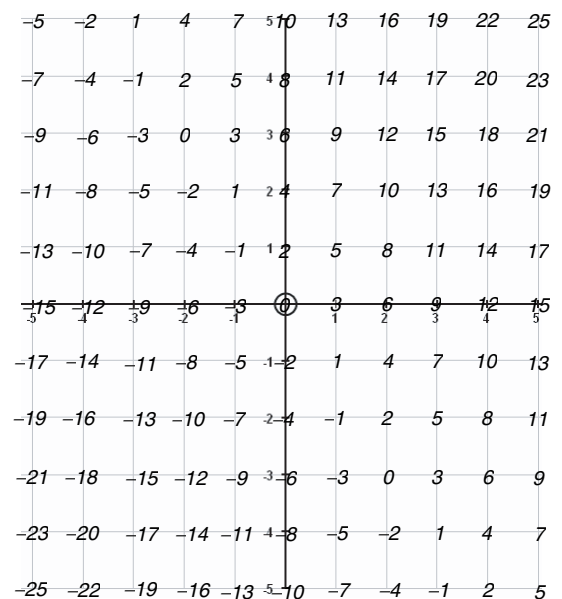

Fig. 2

To clarify, this figure represents the 121 possible linear equations in x and y containing integer coefficients from -5 to 5 that have the solutions $x = 3$ and $y = 2$. The values *along the axes* represent linear equations in *one* variable and are the multiples of the solution values. Numbers lying along any straight line passing through the origin are also always multiples. The numbers in the columns and rows form arithmetic sequences. It is easy from a diagram like this to see the values of x and y that are embedded in it; the x-value of 3 is the common difference along the rows and the y-value of 2 is the common difference along the columns. Generating diagrams like this for different values of x and y and justifying the patterns produced can be instructive.

Using the Diagrams

Regarding this diagram as an 'example space' of equations (Watson and Mason, 2005), *solving* a pair of simultaneous equations means beginning with *just two* of these numbers and determining the values of x and y. In easy cases this can be done at a glance. In Figure 3 we have represented the same two equations $x + y = 5$ and $x - y = 1$ and the vertical difference ($2y$) is clearly 4, so $y = 2$. It is then clear, from looking at either point, that x must be 3. Equations like this, in which x may be eliminated by subtraction, always correspond to two points on a vertical line. Similarly, cases where y may be eliminated by subtraction correspond to two points on a *horizontal* line, and are solved with similar ease.
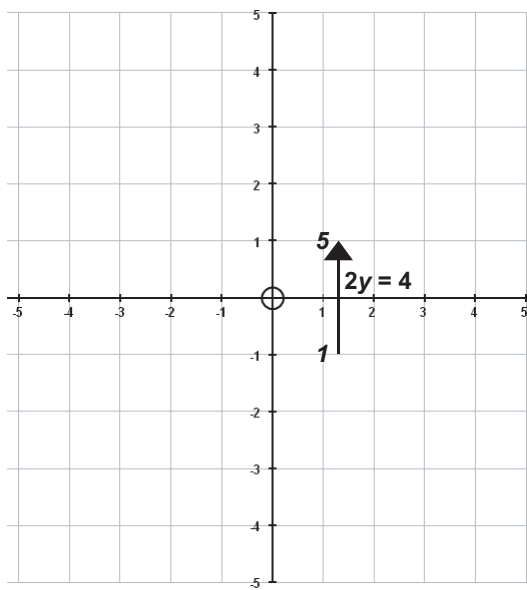

Fig. 3

In more complicated situations, the game is to move one or both of the points until they lie along a horizontal or vertical line. Consider the equations $x + 2y = 7$ and $2x - y = 4$. We plot the values 7 and 4 at the appropriate points and then draw a line passing through the origin and one of our points, say that for the first equation at (1, 2). We can reproduce as many values as we like along these lines by simple scaling up [see Figure 4, where we have added the value 14 at (2, 4)]. It is useful to fill in values when they are vertically or horizontally aligned with the other point. In this case, the value 14 is vertically above the 4, and this leads to $5y = 10$, so $y = 2$, and hence the value of x. An alternative approach puts the value −8 at (−4, 2), leading to horizontal alignment and $5x = 15$, so $x = 3$, from which y can be found.

I am not suggesting that diagrams like these are necessarily a fast and efficient method of solution of simultaneous equations, any more than is the graphical method in Figure 1, although with practice they are not particularly slow. But I think they can provide a helpful insight into the topic and a starting point for further investigation. Too often, simultaneous equations is a topic pupils can be taught to 'do' mechanically and successfully, sometimes without even realizing why 'adding two equations together', for instance, is a meaningful or valid thing to be doing. So I am interested in ways of looking at the problem that shed some light on what is going on.
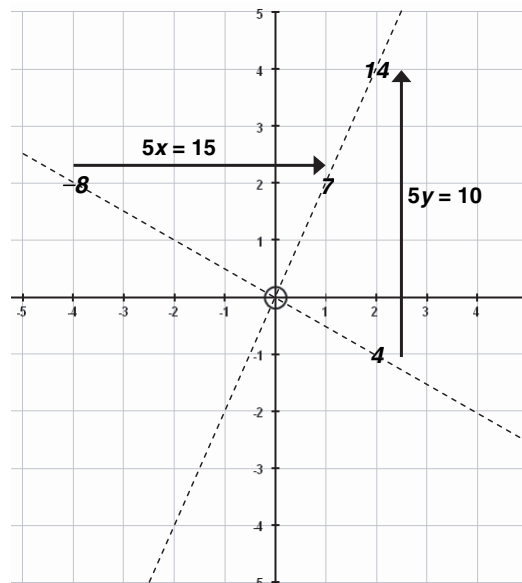

Fig. 4

(Note that in this case you would need to draw *only one* of the dashed lines and calculate *one* new point, not two. Both are included here to show the two alternative methods of solution.)

Further directions for investigation are to ask what pictures are produced when we have *inconsistent equations*, such as $x + y = 7$ and $3x + 3y = 20$? Or when we have *insufficient information*, such as $x + y = 7$ and $3x + 3y = 21$? How is it possible to represent the 'elimination by addition' method of solving equations such as $x + y = 5$ and $2x - y = 4$? See Note 2.

Notes

1. I would guard against thoughtlessly labelling the axes 'x' and 'y', as there is potential for confusion here. One possibility is to write '$ax + by$' as a heading (referring to the values in the plane) and then label the axes a and b, or use "number of xs" and "number of ys" or 'coefficient of x', 'coefficient of y'.

2. It is, of course, possible to do this by effectively multiplying one of the equations by −1. So for these two equations we have 5 at (1, 1) and 4 at (2, −1). By moving the first point to 10 at (2, 2) we obtain $3y = 6$ and hence the solution (this is like doubling the first equation and subtracting the second). However (returning to the original problem), by moving the *second* point to −4 at (−2, 1) we obtain $3x = 9$ and hence the solution, and this is like multiplying the second equation by −1 and subtracting, or alternatively it is like adding the two original equations.

References

Foster, C. 2003 *Instant Maths Ideas for Key Stage 3 Teachers: Number and Algebra*, Nelson Thornes.
Watson, A. and Mason, J. 2005 *Mathematics as a Constructive Activity*, Erlbaum.

Author
Colin Foster, King Henry VIII School, Warwick Road, Coventry CV3 6AQ.
e-mail: **colin@foster77.co.uk**