Questions Pupils Ask!

A ‘Straight’ Question

by **Colin Foster**

The question came from a Year 8 pupil. We were working on the ‘two-dice’ problem, considering the probability of getting different total scores with two ordinary unbiased dice, and had found the sample space diagram given in Figure 1. Pupils had drawn graphs to illustrate the probability distribution, and while some had done bar charts or vertical line graphs, most had joined the points as shown in Figure 2.

	1	**2**	**3**	**4**	**5**	**6**
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Fig. 1 Sample space diagram for the total score of two ordinary unbiased dice

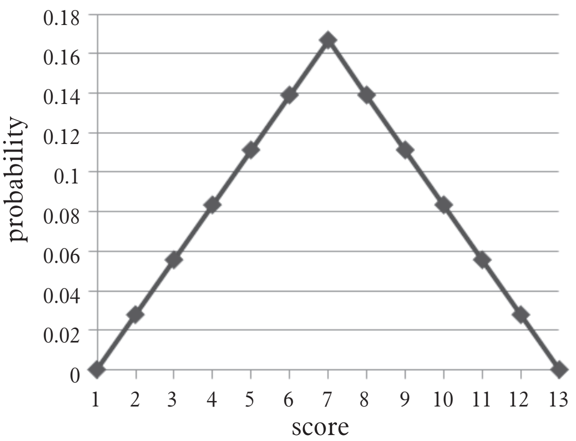

Fig. 2 The probability distribution for the total score with two ordinary unbiased dice

Several pupils expressed surprise that the sides of the graph were straight line segments.

Pupil: “Why is it *straight* up and down?”

Me: “Why *shouldn’t* it be straight?”

Pupils seemed troubled by the shape of the graph. They were familiar with straight line graphs of the forms $x = k$ and $y = mx + c$, which they knew were lines that went on to infinity in both directions. They were also familiar with the more haphazard graphs to be expected in statistics when using real-life data. Prior to our calculations, pupils had reasoned that there were more ways of making a score ‘in the middle’ and only one way of making 2 or 12, so had expected a bell curve (Fig. 3). Straight lines didn’t fit with their perceptions of how the real world behaves. “You don’t get sharp corners in graphs from real life.”

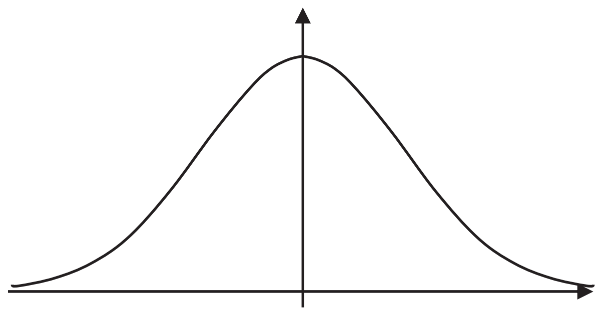

Fig. 3 A bell curve

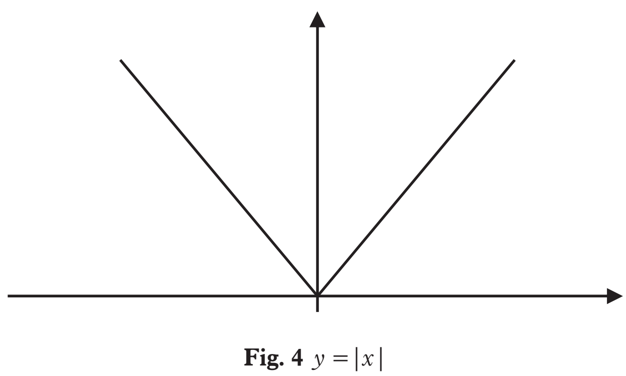

Fig. 4 $y = |x|$

For me, these responses paralleled those of mathematicians in the 19th century during the debates over what constituted a function. For example, it is unlikely that Euler would have accepted $y = |x|$ (Fig. 4) as a function, even if it had been written as a 'nice formula' such as $y = \sqrt{x^2}$, since although it is continuous it is not smooth or differentiable at the origin (see Note 1). Similarly, my pupils did not like a graph with a 'corner' – it seemed too sudden a change for anything in the real world. When we had drawn idealized distance–time and velocity–time graphs, they had argued about how vehicles could suddenly and immediately change direction or velocity and had wanted, quite reasonably, to 'round off' the sharp corners in such graphs. This had reminded me of the Aristotelian view of motion, which thought of straight lines and circles as 'perfect' and had insisted that the path of a projectile consisted of two straight line segments – one up and one down – rather than a parabola, as Galileo showed to be the case (Fig. 5). My pupils had the opposite preconception – correct in this case – that straight lines with a sharp corner would be implausibly 'unnatural'.

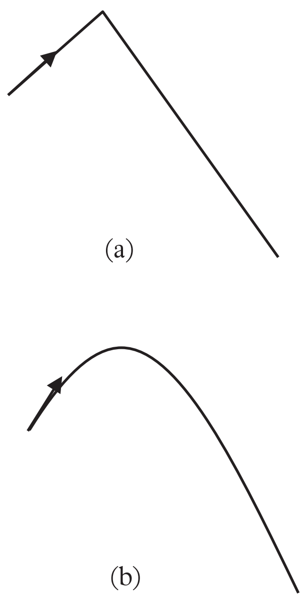

Fig. 5 The path of a projectile according to Aristotle (a) and Galileo (b)

I began to think about what happened with different numbers of dice. With just one die, of course, the uniform distribution ($p(x) = \frac{1}{6}$, $x = 1, 2, 3, 4, 5, 6$) is obtained which, although a straight line, was acceptable to the pupils, since it had no sudden changes (no changes at all, in fact). I naturally wondered what happened with more than two dice, so I calculated the probability distributions for 3, 4, 5 and 6 dice (Table 1) and drew the graphs for 1 to 6 dice (Fig. 6) (see Note 2).

I had never seen these graphs before and was intrigued by many features. The graphs clearly become more 'normal' (in both senses) as the number of dice increases. In one sense, the two-dice graph fits the pattern beautifully, but in another it is an anomaly, since the other graphs clearly appear as smooth curves (see Note 3).

Table 1 Probabilities for the total scores when throwing 1 – 6 ordinary dice

	Total number of dice					
Total score	**1**	**2**	**3**	**4**	**5**	**6**
1	$\frac{1}{6}$	0	0	0	0	0
2	$\frac{1}{6}$	$\frac{1}{36}$	0	0	0	0
3	$\frac{1}{6}$	$\frac{1}{18}$	$\frac{1}{216}$	0	0	0
4	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{72}$	$\frac{1}{1296}$	0	0
5	$\frac{1}{6}$	$\frac{1}{9}$	$\frac{1}{36}$	$\frac{1}{324}$	$\frac{1}{7776}$	0
6	$\frac{1}{6}$	$\frac{5}{36}$	$\frac{5}{108}$	$\frac{5}{648}$	$\frac{5}{7776}$	$\frac{1}{46\,656}$
7	0	$\frac{1}{6}$	$\frac{5}{72}$	$\frac{5}{324}$	$\frac{5}{2592}$	$\frac{1}{7776}$
8	0	$\frac{5}{36}$	$\frac{7}{72}$	$\frac{35}{1296}$	$\frac{35}{7776}$	$\frac{7}{15\,552}$
9	0	$\frac{1}{9}$	$\frac{25}{216}$	$\frac{7}{162}$	$\frac{35}{3888}$	$\frac{7}{5832}$
10	0	$\frac{1}{12}$	$\frac{1}{8}$	$\frac{5}{81}$	$\frac{7}{432}$	$\frac{7}{2592}$
11	0	$\frac{1}{18}$	$\frac{1}{8}$	$\frac{13}{162}$	$\frac{205}{7776}$	$\frac{7}{1296}$
12	0	$\frac{1}{36}$	$\frac{25}{216}$	$\frac{125}{1296}$	$\frac{305}{7776}$	$\frac{19}{1944}$
13	0	0	$\frac{7}{72}$	$\frac{35}{324}$	$\frac{35}{648}$	$\frac{7}{432}$
14	0	0	$\frac{5}{72}$	$\frac{73}{648}$	$\frac{5}{72}$	$\frac{43}{1728}$
15	0	0	$\frac{5}{108}$	$\frac{35}{324}$	$\frac{217}{2592}$	$\frac{833}{23\,328}$
16	0	0	$\frac{1}{36}$	$\frac{125}{1296}$	$\frac{245}{2592}$	$\frac{749}{15\,552}$
17	0	0	$\frac{1}{72}$	$\frac{13}{162}$	$\frac{65}{648}$	$\frac{119}{1944}$
18	0	0	$\frac{1}{216}$	$\frac{5}{81}$	$\frac{65}{648}$	$\frac{3431}{46\,656}$
19	0	0	0	$\frac{7}{162}$	$\frac{245}{2592}$	$\frac{217}{2592}$
20	0	0	0	$\frac{35}{1296}$	$\frac{217}{2592}$	$\frac{469}{5184}$
21	0	0	0	$\frac{5}{324}$	$\frac{5}{72}$	$\frac{361}{3888}$
22	0	0	0	$\frac{5}{648}$	$\frac{35}{648}$	$\frac{469}{5184}$
23	0	0	0	$\frac{1}{324}$	$\frac{305}{7776}$	$\frac{217}{2592}$
24	0	0	0	$\frac{1}{1296}$	$\frac{205}{7776}$	$\frac{3431}{46\,656}$
25	0	0	0	0	$\frac{7}{432}$	$\frac{119}{1944}$
26	0	0	0	0	$\frac{35}{3888}$	$\frac{749}{15\,552}$
27	0	0	0	0	$\frac{35}{7776}$	$\frac{833}{23\,328}$
28	0	0	0	0	$\frac{5}{2592}$	$\frac{43}{1728}$
29	0	0	0	0	$\frac{5}{7776}$	$\frac{7}{432}$
30	0	0	0	0	$\frac{1}{7776}$	$\frac{19}{1944}$
31	0	0	0	0	0	$\frac{7}{1296}$
32	0	0	0	0	0	$\frac{7}{2592}$
33	0	0	0	0	0	$\frac{7}{5832}$
34	0	0	0	0	0	$\frac{7}{15\,552}$
35	0	0	0	0	0	$\frac{1}{7776}$
36	0	0	0	0	0	$\frac{1}{46\,656}$

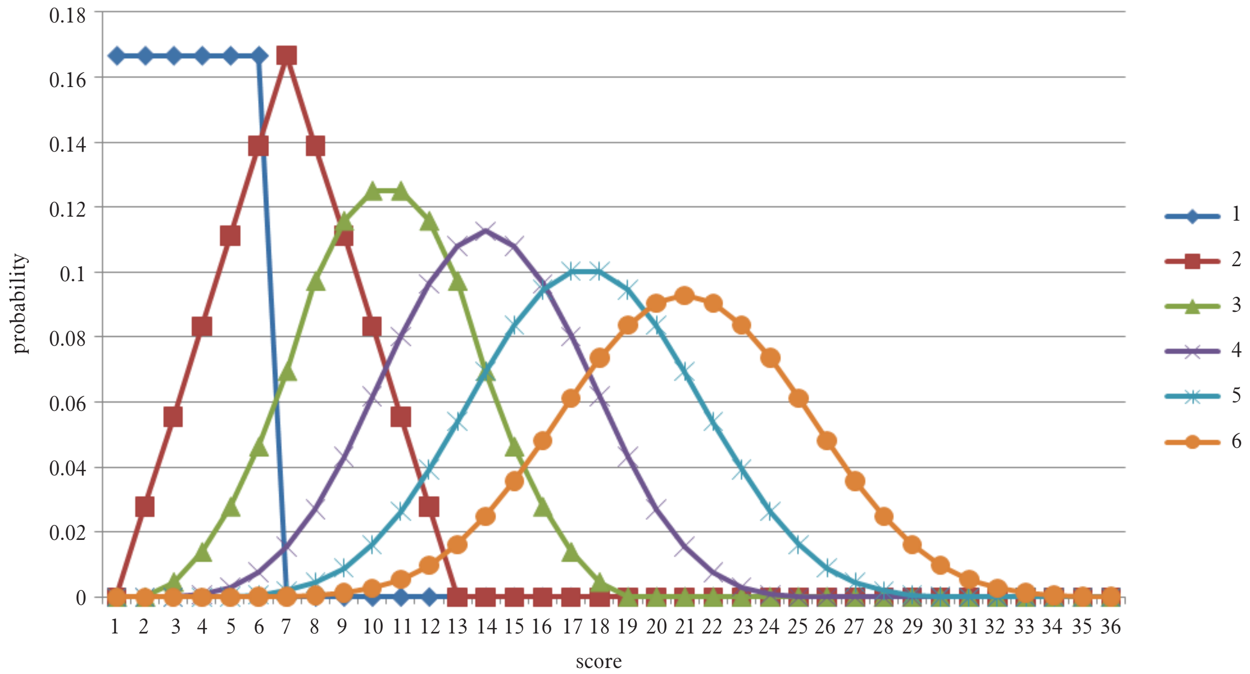

Fig. 6 Probability distributions for the total score with 1–6 ordinary dice

3	4	5	6	7	8
4	5	6	7	8	9
5	6	7	8	9	10
6	7	8	9	10	11
7	8	9	10	11	12
8	9	10	11	12	13

4	5	6	7	8	9
5	6	7	8	9	10
6	7	8	9	10	11
7	8	9	10	11	12
8	9	10	11	12	13
9	10	11	12	13	14

5	6	7	8	9	10
6	7	8	9	10	11
7	8	9	10	11	12
8	9	10	11	12	13
9	10	11	12	13	14
10	11	12	13	14	15

6	7	8	9	10	11
7	8	9	10	11	12
8	9	10	11	12	13
9	10	11	12	13	14
10	11	12	13	14	15
11	12	13	14	15	16

7	8	9	10	11	12
8	9	10	11	12	13
9	10	11	12	13	14
10	11	12	13	14	15
11	12	13	14	15	16
12	13	14	15	16	17

8	9	10	11	12	13
9	10	11	12	13	14
10	11	12	13	14	15
11	12	13	14	15	16
12	13	14	15	16	17
13	14	15	16	17	18

Fig. 7 The 6 layers of a 6 × 6 × 6 sample space for 3 dice with total scores of 6 shown shaded

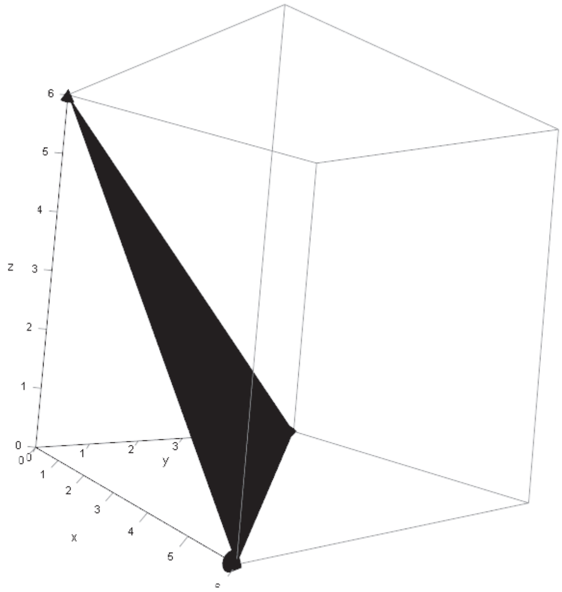

Fig. 8 Ways of getting a score of 6 with three dice

It is easy to see in the sample space diagram in Figure 1 that, for scores 2 to 7, each time you increase the score by 1 you increase the number of ways of achieving it by 1, leading to a line with gradient $\frac{1}{36}$. Similarly, from 7 to 12 we obtain a line with gradient $-\frac{1}{36}$. With three dice, the sample space diagram is three-dimensional, but we can represent it as six two-dimensional layers (Fig. 7).

It is clear that for a total score of 6, say, the number of ways will be a triangle number: 4 + 3 + 2 + 1 = 10, leading to a probability of $\frac{10}{216} = \frac{5}{108}$. Since the triangle numbers increase quadratically, this is going to lead to a curve. Looking at the sample space in three dimensions (Fig. 8), we can see that this pattern will continue for scores up to 6 but then will go awry because, geometrically, the plane will poke out of the faces of the 6 × 6 × 6 cube.

This question led me to compare the sorts of graphs that pupils encounter in different areas of school mathematics at about the same time and to consider how the experiences they acquire in one topic might affect their responses to work in another. Perhaps too often we think of mathematical topics in isolation from one another, yet we know that pupils bring to their work not only their experiences from outside of the classroom but also those from other mathematical areas that 'we are not doing today'.

Notes

1. For an interesting discussion of this, see chapter 6 of Penrose, R. (2005) *The Road to Reality*, Vintage.
2. I calculated these using the probability generating function $\left(\frac{x(1-x^6)}{6(1-x)}\right)^n$, where n is the number of dice and x is a dummy variable. I used computer algebra software to expand these for the different values of n and then wrote down the coefficients of x in the expansions.
3. Of course, this is to some extent in the eye of the beholder, since all of these graphs consist of a finite collection of points and would be drawn as vertical line graphs were it not that this would make it harder to compare them on one pair of axes.

Keywords: Curves; Dice; Graphs; Probability distribution.

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