A quick informal survey among colleagues revealed a number of different approaches to finding an expression for the \( n \)th term of a sequence of numbers. We give a lot of attention to this particular skill because of its usefulness in the GCSE coursework investigation, so it seemed worth considering the pros and cons of the different methods.

**Method 1. Inspection**

The quickest approach to many sequences, though the hardest to learn, is the “Ah, I see what it is!” method! Sometimes the clue comes from the source of the sequence, perhaps the geometry of a pattern, for example. Sometimes, it is possible to see how numbers with not too many factors might have been produced.

For example, in this sequence, 15 suggests itself as \( 3 \times 5 \), and 35 as \( 5 \times 7 \). Checking this pattern for the other terms quickly confirms that \( u = (n+1)(n+3) = n^2 + 4n + 3 \).

Pupils frequently need to be encouraged that inspection is a ‘proper method’ and not ‘cheating’, although they’ll need to explain that it wasn’t simply inspection of someone else’s work!

**Method 2. Simultaneous Equations**

All the other methods begin by finding the difference between successive values, and then the differences between those successive differences, and so on until a row of differences turn out to be constant. For example,

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( u )</td>
<td>8</td>
<td>15</td>
<td>24</td>
<td>35</td>
<td>48</td>
<td></td>
</tr>
</tbody>
</table>

1st differences: 7, 9, 11, 13
2nd differences: 2, 2

If the 1st row of differences turn out to be the same, the sequence is 1st degree (linear): \( u = a + bn \).

If the 2nd row of differences turn out to be the same (as above), the sequence is 2nd degree (quadratic): \( u = a + bn + cn^2 \).

If the 3rd row of differences turn out to be the same, the sequence is 3rd degree (cubic): \( u = a + bn + cn^2 + dn^3 \).

And so on, where \( a, b, c, d, \) etc. are all constants to be determined. (The final constant in each equation can’t be zero, otherwise the sequence wouldn’t have the required degree, but some or all of the other constants could be zero.)

The simultaneous equations method starts with the appropriate formula (quadratic for our example), and with three unknowns to find \( (a, b, c) \) substitutes \( n \) and \( u \) values for the first three terms, as below.

\[
\begin{align*}
  u &= a + bn + cn^2 \\
  n &= 1 \quad u &= 8 = a + b + c \\
  n &= 2 \quad 15 = a + 2b + 4c \\
  n &= 3 \quad 24 = a + 3b + 9c
\end{align*}
\]

Then the task is to solve these simultaneous equations to find \( a, b \) and \( c \) and hence the formula for \( u \).

The main difficulty with this method is solving these simultaneous equations, which for a cubic (or higher) sequence can be heavy work. GCSE pupils are not normally expected to solve simultaneous equations in more than two unknowns, and three is necessary even for a quadratic sequence. For that reason I don’t generally encourage pupils down this route.

**Method 3. Comparison with the General Case**

This begins by finding differences until a row of constant differences are obtained (as above) so that the degree of \( u \) is found. Then the differences method is applied to the general case of that degree, as below for a quadratic sequence \( u = a + bn + cn^2 \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( u )</td>
<td>( a )</td>
<td>( a+b+c )</td>
<td>( a+2b+4c )</td>
<td>( a+3b+9c )</td>
</tr>
</tbody>
</table>

1st differences: \( b+3c \), \( b+5c \), \( b+7c \)
2nd differences: \( 2c \), \( 2c \)

By comparing the numbers in the differences table for our sequence with the expressions in this table we can say that

\[
\begin{align*}
  2c &= 2 \Rightarrow c = 1 \\
  b + 2c &= 7 \Rightarrow b + 3 \times 1 = 7 \Rightarrow b = 4 \\
  a + b + c &= 8 \Rightarrow a + 4 + 1 = 8 \Rightarrow a = 3
\end{align*}
\]
Differences over Differences  
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1st differences

for, working from the bottom line upwards.)

column for table is extended to the left to find the 0th term and a
difficulty.
The advantage of this type of formula, is that substituting

2nd differences

Again it is possible to use this method with a differences
table that goes back to \( n = 0 \) (see Method 3). Then the most
convenient formula is \( u = A + Bn + Cn(n-1) \) for quadratic,
with similar formulas for cubic or higher sequences.

Method 5. Reducing the Degree by One

A long-winded but easy-to-understand method is first to
find the coefficient of the highest power of \( n \) and then
evaluate and subtract that term from each term of the
original sequence to leave a sequence that is at least one
degree less.

For example, constant 2nd differences of 2 tell us that in
\( u = a + bn + cn^2 \) the value of \( c \) is 1. So we work out \( cn^2 (= n^2 \)
here) for each value of \( n \) and subtract that much from each
corresponding value of \( u \).

The new sequence for \( u – n^2 \) is linear, and by differences
or inspection the \( n^n \) term is 4\( n \), so \( u – n^2 = 4n + 3 \) or
\( u = n^2 + 4n + 3 \).

The logic of this can be appealing, but it is too lengthy for
cubic or higher sequences, particularly if some of the
numbers are not integers.

Conclusions

For myself, I firmly reject Method 2. Method 5 may be
helpful for understanding but is too inefficient for general
use. Method 4 may be quickest if all you need is, say, the
100th term, because you can substitute \( n = 100 \) without
needing to expand the brackets and simplify. But if you want
an answer in the form \( u = a + bn + cn^2 + \ldots \) I don’t think
you can beat Method 3.

That’s just my conclusion. I’d be very interested to hear
other readers’ opinions. 

Reference

Wakefield, P. 2003 ‘Newton’s Forgotten Formula’, Mathematics in School,
32. 2.

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