Fitting shapes inside shapes: Closed but provocative questions

by **Colin Foster**

Mathematics teachers are often encouraged to try to turn closed questions (such as 'What is 8 × 3?') into open questions (such as 'What numbers multiply to make 24?') because open questions are widely perceived to be richer and more productive. However, sometimes a question that is technically closed – even a dichotomous yes/no question – can lead to lots of interesting discussion and thought. I call such questions *closed but provocative* (Foster, 2015). In real life a lot can ride on a closed question – 'Do you find the defendant guilty or not guilty?' Closed questions are not necessarily trivial; indeed, many of the great unsolved problems in mathematics, such as the Riemann hypothesis (Sabbagh, 2003), boil down to closed questions. Is the Riemann hypothesis true or false? The answer is either yes or no (Note 1).

Also at the level of school mathematics, the power of *closed but provocative* questions should not be underestimated. Recently in *MiS* Chris Pritchard has given us a fascinating series of articles on fitting shapes inside shapes. (You will find them in the issues for November 2010 through to September 2013, with problem sets following to January 2015.) Continuing this theme, I offer here three questions – all of them closed – which I hope that you may find provocative:

1. Will a 1 × 6 rectangle fit completely inside a 5 × 5 square?
2. Will a 2 × 13 rectangle fit completely inside a 9 × 12 rectangle?
3. Will a 1 × 4 × 8 cuboid fit completely inside a 6 × 6 × 6 cube?

Naturally, you are not allowed to break up the shapes in any way!

Of course, there is an implicit invitation in these closed questions to justify your answers as well as to generalize and invent 'easy' and 'hard' problems along these lines.

1. Will a 1 × 6 rectangle fit completely inside a 5 × 5 square?

A convenient way to explore problems of this kind is to slide a transparency of centimetre squares over the top of a centimetre-square grid. A first go at a scale drawing (Fig. 1) is inconclusive. It would appear that the 1 × 6 rectangle will either just fit or just not fit, but it appears to be too close to call. Of course, that is why I selected this example, in the hope that the uncertainty would be provocative! For me there is something intriguing about a yes/no question that can be simply stated, and which everyone can immediately understand, but which is not quick or easy to answer.

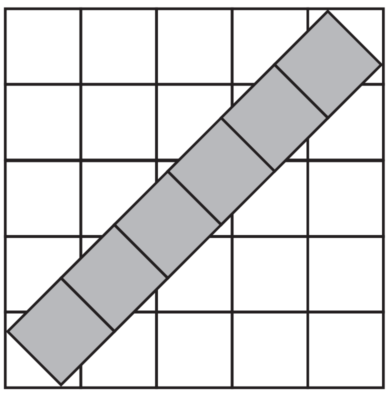

Fig. 1 Will a 1 × 6 rectangle fit completely inside a 5 × 5 square?

Sometimes students are given exercises in which they are asked to calculate lengths which they might have found it easier to draw accurately and measure, so it is nice sometimes to have situations where accurate drawing is just not accurate enough and calculation is essential. When carrying out calculations in exercises, students frequently ask how much accuracy they should give in their answers, and this question may be taken as a sign that the values that they are obtaining are not being used for any wider purpose. It is impossible to say what 'reasonable' accuracy is without knowing for what 'reason' the calculation is being carried out. Sadly for students, the answer is often 'no reason at all'. In the problems we are thinking about here, it is obvious that you need to be accurate enough to answer the question and that determines how accurate is accurate 'enough'.

We can tackle the first problem in the general case of a square with sides of length s and a rectangle with sides a and b. We can assume, without loss of generality, that $a > b$. If $a \le s$, then the solution is trivial – the rectangle can be placed with its sides parallel to the sides of the square. So for an interesting problem we must have $a > s$. Note that we cannot *also* have $b > s$, otherwise $ab > s^2$, and the area of the rectangle would be larger than the area of the square – and a larger area cannot be contained within a smaller one. So this means that for our problem we must have $a > s > b$.

We will assume that the optimal position for the rectangle is with the rectangle's longer line of symmetry lying along the diagonal of the square (shown dashed in Figure 2). It would appear that we can make a longer at the expense of making b shorter, and the rectangle will still fit inside the square. So we will assume fixed s and a and try to find the maximum possible b.

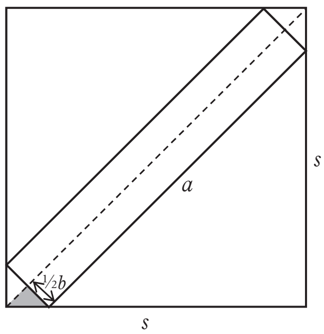

Fig. 2 An $a \times b$ rectangle inside an $s \times s$ square

This turns out to be quite easy once we observe that, because of the 45° angles, the shaded right-angled triangle in Figure 2 is isosceles, from which we can see that the dashed diagonal of the square has length $a + b$, giving the condition, $a + b \le \sqrt{2}s$; or $b \le \sqrt{2}s - a$.

For our 1×6 rectangle and the 5×5 square, we have $\sqrt{2}s - a = 5\sqrt{2} - 6 = 1.071\ldots$, which means that $b = 1$ satisfies the inequality and a 1×6 rectangle *does* fit inside a 5×5 square.

2. Will a 2 × 13 rectangle fit completely inside a 9 × 12 rectangle?

Let's attempt to generalize this question to fitting an $a \times b$ rectangle inside a $c \times d$ rectangle. In a similar way to before, we can assume without loss of generality that $a > b$ and $c > d$. If $a < c$ and $b < d$, the small rectangle will easily fit inside the larger one by placing its sides parallel to the sides of the larger rectangle, so this situation is not very demanding. If $a > c$ and $b > d$, the 'smaller' rectangle will have a larger area than the 'larger' one, which is impossible. This means that the only two cases we need to consider are (i) $a > c$ and $b < d$ and (ii) $a < c$ and $b > d$. However, these are symmetrical, so without loss of generality we can assume that $a > c$ and $b < d$, leaving just the case $a > c > d > b$.

As before, we will assume fixed a, c and d and try to find the maximum possible b. We will assume that the best position for fitting the largest possible $a \times b$ rectangle inside the $c \times d$ rectangle is where all four vertices of the $a \times b$ rectangle lie on the edges of the $c \times d$ rectangle (Fig. 3).

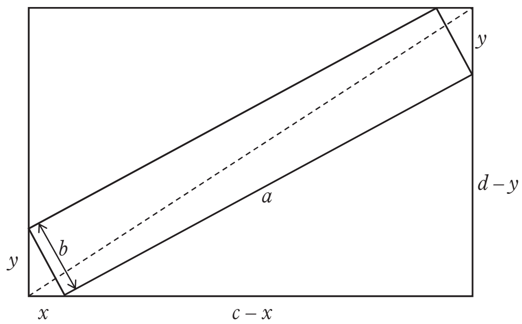

Fig. 3 An $a \times b$ rectangle inside a $c \times d$ rectangle

By similar triangles,

$$\frac{y}{b} = \frac{c - x}{a} \text{ and } \frac{x}{b} = \frac{d - y}{a}, \text{ giving}$$

$$ax + by = bd$$

$$bx + ay = bc,$$

so $x = \dfrac{b(ad - bc)}{a^2 - b^2}$ and $y = \dfrac{b(ac - bd)}{a^2 - b^2}$, remembering that since $a > b$ the denominators cannot be zero.

Now, using Pythagoras' Theorem in the bottom-left right-angled triangle,

$$x^2 + y^2 = b^2$$

$$\frac{b(ad - bc)^2}{(a^2 - b^2)^2} + \frac{b^2(ac - bd)^2}{(a^2 - b^2)^2} = b^2.$$

So, $(ad - bc)^2 + (ac - bd)^2 = (a^2 - b^2)^2$ (*)

meaning that for a given a, c and d, b cannot be larger than the value of b that satisfies this equation (Note 2).

As a check, we can let the $c \times d$ rectangle be a square by setting $c = d = s$, giving

$$(a - b)^2 s^2 + (a - b)^2 s^2 = (a - b)^2 (a + b)^2, \text{ and,}$$

since $a \neq b$, this reduces to $2s^2 = (a+b)^2$ or $a+b = \sqrt{2}s$, since $a > b > 0$, as we obtained for our first problem type.

We can now answer our question about whether a 2×13 rectangle will fit completely inside a 9×12 rectangle by substituting $a = 13, c = 12$ and $d = 9$ into (*) and solving for b, which gives $b = 2.1410\ldots$, meaning that, indeed, a 2×13 rectangle *will* fit inside a 9×12 rectangle.

3. Will a $1 \times 4 \times 8$ cuboid fit completely inside a $6 \times 6 \times 6$ cube?

This is hard, and I will leave it for the reader to ponder!

Notes

1. Well, unless you entertain the thought that it might be 'undecidable'. Ian Stewart says that he would "be surprised if the Riemann hypothesis were like that, and amazed if anyone could prove it to be undecidable even if it were" (Stewart, 2013, p. 278).
2. We are ignoring the possibility that there might be more than one solution for b – or none.

Acknowledgement

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References in addition to the 'pegs series'

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