Non-Linear Inequalities

by **Colin Foster**

It is normal for Year 10–11 pupils to be asked to shade inequalities of the forms

$$x > c,$$
$$y > bx + c,$$
$$ax + by > c$$

and possibly $y > (x - a)(x - b)$,

where a, b and c are constants, with $>$, $<$, $\geq$ or $\leq$ signs. But they are unlikely to encounter other kinds of non-linear inequalities besides quadratics. This seems a pity, as there is some interesting thinking involved in sketching them. It is one of those tasks that requires no extra knowledge—no 'teaching', as such—and can be fun for pupils to experiment with. It can also produce some surprising results—several of them certainly surprised me!

For example, try this:

> Shade the inequality $\frac{1}{y} > \frac{1}{x}$.

You might initially think that it is equivalent to $x > y$, perhaps excepting when x or y is zero. I remember when I was at school understanding that inequality signs reverse direction when you multiply or divide both sides by a negative number. This was 'obvious', because although $3 > 2$, it was clear that $-3 < -2$. Unfortunately, this 'proof by one example' was also sufficient to convince me that the inequality sign would reverse when 'reciprocalling' both sides of an inequality, because similarly if $3 > 2$ then $\frac{1}{3} < \frac{1}{2}$. What I don't think I had thought about until much later was that if $3 > -2$ then it does *not* follow that $\frac{1}{3} < -\frac{1}{2}$. Sometimes it is hard to be sure when 'proof by generic example' is really a proof and when there may be something important that you haven't considered. These remarks may or may not provoke you to look again at your shading of the above inequality (perhaps checking at some 'test points') before reading on. Be warned that graph-drawing software will not necessarily get it right!

Here are some more that you might like to try:

> Shade the inequalities:
>
> $y^2 > x^2$ $\quad\quad$ $\frac{x}{y} > 1$
>
> $\frac{1}{y^2} > \frac{1}{x^2}$ $\quad\quad$ $\frac{x}{y} > \frac{y}{x}$
>
> $\frac{1}{x+y} > \frac{1}{x}$ $\quad\quad$ $\frac{1}{x-y} > \frac{1}{x+y}$
>
> $\frac{1}{x-y} > \frac{1}{x}$ $\quad\quad$ $\frac{x+y}{x-y} > 0$
>
> Do you expect in advance any of these to look the same as each other?
>
> Make up some more like these and shade them.

There are different ways in which you might do these. The inequality $\frac{1}{y} > \frac{1}{x}$ is indeed equivalent to $x > y$ if x and y are both positive or both negative. In those cases, xy is positive, and multiplying both sides by a positive quantity leaves the inequality sign the same way round. So this sorts out the first and third quadrants (Fig. 1) (Note 1). On the other hand, if one of x or y is negative

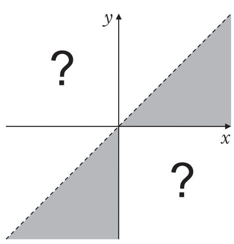

Fig. 1 First and third quadrants correct

and the other positive, then the inequality sign will *reverse* when we multiply by xy, so we will have $x < y$ in the second and fourth quadrants. But $x < y$ *everywhere* in the second quadrant and *nowhere* in the fourth, leading to the result shaded in Figure 2. It may look a bit odd having disconnected bits of shading, but checking the coordinates of some particular points will verify that each region is correct.

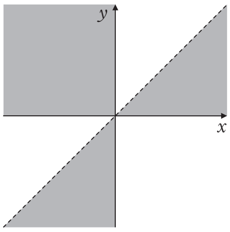

Fig. 2 $\frac{1}{y} > \frac{1}{x}$

Alternatively, we can write:

$$\frac{1}{y} > \frac{1}{x}$$

$$\frac{1}{y} - \frac{1}{x} > 0$$

$$\frac{x - y}{xy} > 0$$

For this quotient to be positive, either the numerator and the denominator must both be positive or they must both be negative, leading to the alternatives:

either $x - y > 0$ *and* $xy > 0$, which means $x > y$ in the first and third quadrants,

or $x - y < 0$ *and* $xy < 0$, which means $x < y$ in the second and fourth quadrants,

as before.

It can also be helpful to observe symmetry and anti-symmetry between the variables x and y. In this case, swapping around x and y precisely *reverses* the inequality, so the finished shading should be anti-symmetrical in the line $y = x$ (i.e. shading reflects into non-shading and vice versa) or, equivalently, symmetrical in the line $y = -x$.

Some of the shadings given in Figure 3, on the next page, correspond to some of the inequalities given in the box above. You might like to find inequalities to represent any that don't and create others to make other similar shadings.

Note

1. Throughout this article, shading shows the regions where the inequality *holds*. I have not attempted to show the exclusion of points or lines which correspond to undefined values resulting from division by zero.

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Patterns in **Expanding** Brackets

The short fillers on pages 14 and 23 have used entries in Pascal's Triangle for elegant applications in numbers and shape. We cannot let algebra slip through Pascal's grid of numbers.

Try this with your Year 7s or 8s.

When we expand the brackets in $(a + b)^2$ we get the expression

$(a + b)^2 = a^2 + 2ab + b^2$.

The numbers in front of a^2, ab and b^2 are 1, 2 and 1. These are called the *coefficients of the expansion*.

Expand the following brackets and write down the coefficients of each expansion.

$(a + b)^3 =$

$(a + b)^4 =$

$(a + b)^5 =$

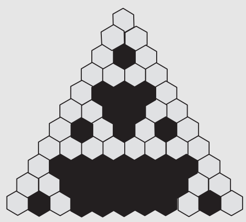
$(a + b)^6 =$

Can you identify a pattern with the coefficients? Use the pattern to suggest the coefficients in the expansion of $(a + b)^7$.

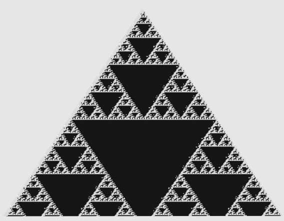
Write down the first four terms in the expansion of $(a + b)^{12}$.

And finally, after all that hard work, colour in Pascal's triangle to produce an example of a fractal.

Colour the evens in one colour and the odds in a different colour.



Keep going with a huge Pascal's triangle and this is what you get.



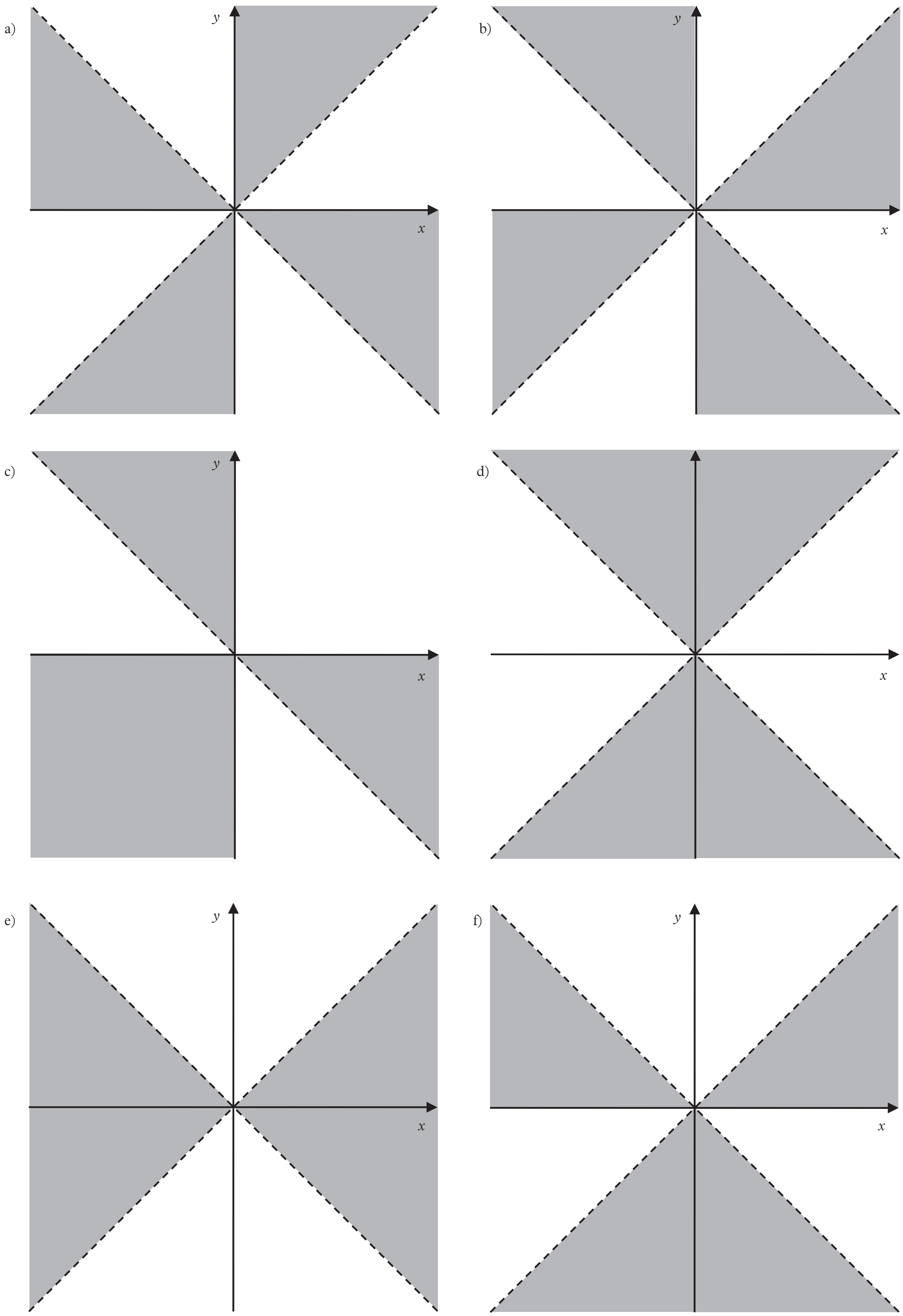


Fig. 3 Six shaded inequalities