Simultaneous Inequalities

by **Colin Foster**

I am thinking of four numbers:

$\boxed{a}$ $\boxed{b}$ $\boxed{c}$ $\boxed{d}$

1. The sum of the first two numbers is greater than the sum of the last two numbers.
2. The sum of the first and third numbers is greater than the sum of the second and fourth numbers.

What can you conclude from this?

You might like to take some time to think about this before reading on.

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You could begin by experimenting to try to find numbers that satisfy the two conditions. Alternatively, using algebra, we have:

$$a + b > c + d \qquad (1)$$

$$a + c > b + d. \qquad (2)$$

These simultaneous inequalities look very like simultaneous equations, so let's proceed as we might with equations. Adding,

$$2a + b + c > b + c + 2d. \qquad (1) + (2)$$

So $a > d$.

Subtracting,

$$b - c > c - b. \qquad (1) - (2)$$

So $b > c$.

But something is wrong here. The numbers 5, 1, 3, 2, for instance, satisfy the two given conditions, but although $a > d$ it is *not* the case that $b > c$. Where have we gone wrong? What should we have done differently?

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For learners from Year 9 upwards who are familiar with solving simultaneous *equations*, simultaneous *inequalities* provides an interesting area to investigate. Provided learners know the meaning of the inequality signs > and <, they do not need to be told or shown anything else in order to explore problems like this for themselves.

In this problem, it *does* follow that $a > d$ but it does *not* follow that $b > c$. In fact, b may be less than c (as seen above), greater than c (e.g. 4, 3, 2, 1), or equal to c (e.g. 3, 2, 2, 1). Adding the inequalities was valid but subtracting them was not, and it will take learners a bit of thought to see why.

Often, when learners are solving simultaneous equations, they are quite hazy on just why it is valid to add or subtract pairs of equations. Is it even obvious that if you add two equations you will get another *equation*? Why don't we end up with something with two equals signs? It is helpful to think of the process as adding the same thing to both sides of *one* of the equations. So if our two equations are

$$A = B \qquad (\star)$$

$$C = D \qquad (\dagger)$$

then, just as it would be valid to add C to both sides of ($\star$) to give

$$A + C = B + C,$$

since we know from (†) that D is equal to C, we can rewrite this as

$$A + C = B + D,$$

which is the result of adding the equations together. A similar argument works for subtracting equations.

But are you allowed to add or subtract pairs of simultaneous *inequalities*? Learners might need to think quite hard about this. We can add inequalities if the inequality signs are the same way round. Learners might be able to explain in words why this is valid. If a bus is heavier than a car, and a dog is heavier than a cat, then a bus-and-a-dog will be heavier than a car-and-a-cat, because the bus by itself outweighs the car and the dog by itself outweighs the cat. Alternatively, inequalities can be written as equations by including a letter such as ε (> 0) to make up the difference between the two sides.

So

$$a + b > c + d \qquad (1)$$

is equivalent to

$$a + b = c + d + \varepsilon_1. \qquad (1_\varepsilon)$$

Similarly, (2) is equivalent to

$$a + c = b + d + \varepsilon_2. \qquad (2_\varepsilon)$$

Since these are *equations*, we can add to give

$$2a + b + c = b + c + 2d + \varepsilon_1 + \varepsilon_2. \qquad (1_\varepsilon) + (2_\varepsilon)$$

Simplifying,

$$2a = 2d + \varepsilon_1 + \varepsilon_2.$$

So

$a = d + ½(\varepsilon_1 + \varepsilon_2)$,

meaning that $a > d$, as we thought.

Although subtracting inequalities may have looked just as plausible when we did it before, it is not valid to subtract inequalities in which the signs are the same way round. We can see why by using the epsilons again.

Subtracting gives

$b - c = c - b + \varepsilon_1 - \varepsilon_2$. $\quad (1_\varepsilon) - (2_\varepsilon)$

Now, although each epsilon is positive, we do not know whether $(\varepsilon_1 - \varepsilon_2)$ is positive, negative or zero, so we cannot convert our final line into a useful inequality. The relation between b and c could be >, < or =.

You can use epsilons to show that subtracting inequalities *is* valid if the inequality signs are *opposite*, so rewriting (2) as

$b + d < a + c \quad (3)$

and subtracting from (1) leads to

$a - d > d - a \quad (1) - (3)$

which *is* valid but just gives us $a > d$ again. So can we say anything more than that $a > d$ from the two given conditions?

There is much for learners to explore here, such as the effects of modifying the given conditions or including a third one such as:

3. The sum of the first and last numbers is greater than the sum of the middle two numbers.

By inventing additional conditions, learners can try to tie down the problem so that there is a unique solution. They can also explore what happens if there are three numbers or five numbers, etc.

Problems like this, in which learners make sense of a novel area of mathematics by using their 'native wit', can be a good way to extend an area of the curriculum. The teacher doesn't need to 'explain' the topic – learners can sort it out for themselves by exploration. In this case, they can get a feel for the problem by experimenting with numbers, and also use specific values to verify the conclusions of their algebra or find counterexamples. There is much scope for creative invention of similar problems.

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