

Slippery Slopes

by Colin Foster

A common way of beginning a mathematics topic is to announce a definition or a result and then ask pupils to work on tasks or exercises that apply what has been offered. I have often found it more interesting to begin the other way round, with a problem that leads to a *need* for definitions or theorems that will help with its solution. Working this way can be more enjoyable and challenging, but is also more dangerous, because the lesson may unfold in unforeseen ways!

I decided to introduce the concept of gradient to my year 8 class by means of the drawing in Figure 1.

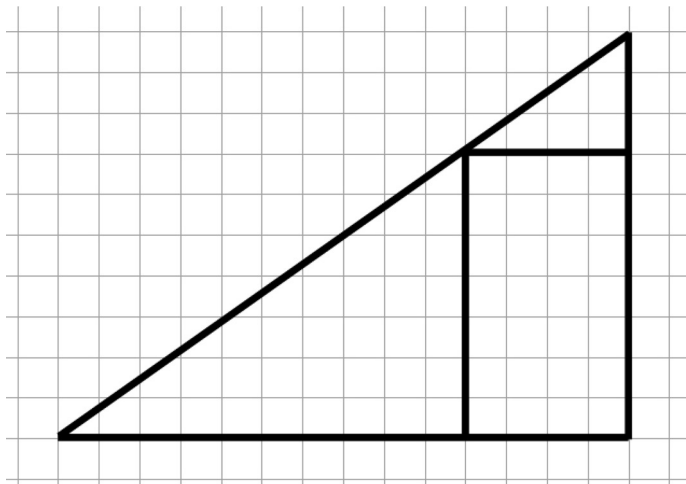


Fig. 1

I asked pupils to calculate some different areas. I didn't state a lesson objective (in this case that would have killed off the lesson!); as far as the pupils were concerned, we were reviewing area calculations. I reassured those who thought that this was too easy – “Trust me; you'll find something interesting!” I knew what I was expecting. The area of the large triangle (the whole shape) comes to $\frac{1}{2} \times 14 \times 10 = 70$ square units. The total area of the two smaller triangles and the rectangle comes to $\frac{1}{2} \times 10 \times 7 + \frac{1}{2} \times 4 \times 3 + 4 \times 7 = 69$ square units.

Everyone noticed the contradiction and most assumed that they had made a mistake, often checking products like 4×7 on a calculator! There was a lot of energy resulting from the need to resolve these conflicting answers. At this stage I mainly listened and encouraged pupils to think out loud. Explanations came quickly: many felt the problem was that the sloping line did not pass exactly through grid points (i.e. it was not a 45° line). This made me worry about their understanding of the area of a triangle as for *any* triangle.

With hindsight, I don't think they necessarily had doubts about the calculation of areas of non-isosceles-triangles: sometimes, any explanation seems better than none, and there was a slightly panicky atmosphere, in which pupils wanted to grasp *some* resolution quickly, even if it wasn't completely convincing. The difference of just 1 square unit led some to suggest 'rounding errors', which is not far off the truth, but didn't see where non-integer answers could come from.

Many pupils were willing to work with the problem in their own way, discussing possible answers with one another. Others were unhappy, but unsure what to do. So I offered two possible ways to proceed:

1. Try making a *really accurate* copy of the drawing on 1 cm squared paper.
2. Look at a similar puzzle (Figure 2) for clues (Wells, 1992). The pieces in the top square have a total area of $8 \times 8 = 64$ square units, whereas the same pieces arranged differently seem to make a rectangle of area $13 \times 5 = 65$ square units. (If you make this one you can cut up the pieces and shuffle them around.)

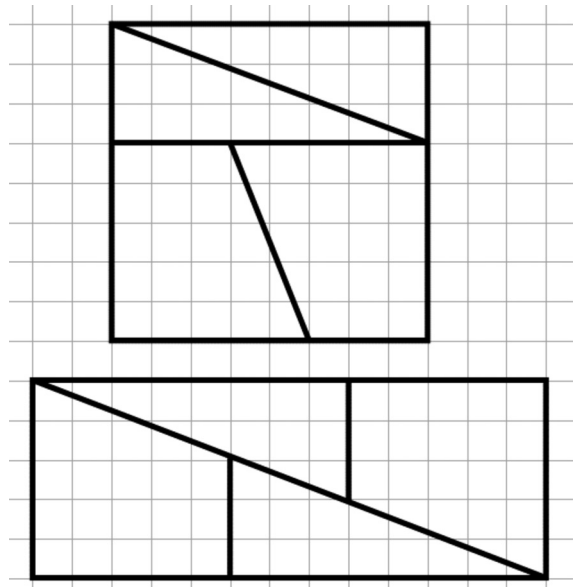


Fig. 2

Sometimes I gave more explicit hints:

Me: Why do you think I made the lines so thick?

Pupil: So they stand out?

Me: Partly, but I'm afraid there's a more sinister reason!

Pupils with better drawing skills (and sharper pencils) began to realize that the thick lines were concealing the fact that the two small triangles in Figure 1 had hypotenuses of different slopes ($\frac{7}{10} = 0.7$ and $\frac{3}{4} = 0.75$), so that a greatly exaggerated drawing might look like Figure 3. The obtuse-angled shaded triangle contains the missing 1 unit of area. (That this area is 1 unit of area actually follows from Pick's theorem, since the vertices lie on grid points and there are none inside or along the edges.)

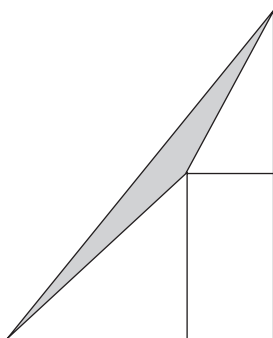


Fig. 3

As pupils tried to explain this (of course, without calculating slopes or using words like 'gradient'), I could introduce the notion of gradient very naturally. A feeling that '10 along and 7 up isn't the same as 4 along and 3 up' could be formalized, and the definition seemed easy and helpful.

These puzzles work when gradients are close but not equal, arising from nearly but not exactly equivalent fractions. (The paradox in Figure 2 relies on the fact that $\frac{3}{8}$ is less than $\frac{2}{5}$, but only slightly.)

I would have liked to look at the coordinates of collinear points, and ask pupils to come up with a method for deciding without drawing whether three points lie in a straight line or not, but there was insufficient time. I encouraged pupils to experiment with pairs of nearly equal fractions like these to try to construct their own puzzles, but it isn't easy! I did feel, though, that the concept of gradient was more firmly understood than when I have taught it by simply stating that "there's this thing called gradient, and this is what it is, and this is how you work it out." Unfortunately, the tendency or requirement to state learning objectives precisely at the beginning of a lesson can make it difficult for concepts to emerge naturally in a problem-solving setting like this. ☒

Reference

This well-known "Chessboard Paradox" goes back to William Hooper (1774) *Rational Recreations*, cited in Wells, D. 1992 *The Penguin Book of Curious and Interesting Puzzles*, Penguin, p. 41. For an animated version, see <http://www.cut-the-knot.org/Curriculum/Fallacies/FibonacciCheat.shtml>. See also Curry's Paradox, <http://www.mathematik.uni-bielefeld.de/~sillke/PUZZLES/jigsaw-paradox.html>

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Edexcel GCSE Statistics

Gill Dyer, Jane Dyer, David Kent, Keith Pledger, Brian Roadnight and Gordon Skipworth
Heinemann 2003
ISBN 0 435 53312 6
330 pages, 264 × 195 mm, softback
£14.99

Two of the authors, namely Gill Dyer and Gordon Skipworth were involved with the writing of Heinemann Modular Mathematics for Edexcel AS and A Level Statistics (S1–S4) textbooks. Their influence can be clearly seen in the style and presentation of the material in this GCSE textbook.

The contents are laid out in ten sections, although the first section is a general introduction. The remaining nine sections match the new requirements of Edexcel GCSE Statistics. There is extensive use of bold, clear colour diagrams and print. Foundation and Higher tier material are clearly differentiated. A red bar, displayed next to the text indicates Higher tier work. Suggested use of ICT in the sections covering Representation and Processing Discrete Data, Scatter Diagrams and Correlation and Time Series should prove to be an interesting and challenging approach

to understanding the work. Section 10 deals with the coursework element of the final assessment. Included are guidelines suggesting helpful steps that could be taken to produce a piece of coursework. There are also suggestions for coursework projects, some of which may need to be treated with caution!

For each section, there are exercises after each set of examples. These vary from 3–12 questions. Teachers should not rely on the examples providing sufficient understanding of a particular concept. Each section includes a Revision exercise (varying from 3–8 questions) and a Summary of Key Points after the Revision exercise. These are a particularly good feature of the textbook and should greatly help to reinforce students' understanding of each section. At the end of the textbook there are two Examination Practice Papers; one is based on Foundation tier and the other on Higher tier. Each paper is comprised of sections A & B with a total of 80 marks for Foundation tier and 100 marks for Higher tier. There is a comprehensive list of answers at the back of the textbook. Perhaps the answers should have been printed as a separate booklet, as there may be great temptation on the part of the students to copy such detailed answers.

Approximately 60% of the textbook covers work at Foundation tier. Most of Section 8 (Probability) and the whole of Section 9 (Probability Distributions) have been written with only Higher tier students in mind. At £14.99 a copy, it seems unlikely that *Edexcel GCSE Statistics* will be bought exclusively for Foundation tier students, but is certainly worthy of serious consideration for Higher tier students.

Tim Sutton-Day

Enhancing Primary Mathematics Teaching

Edited by Ian Thompson
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This book is the third in an important trilogy of edited collections on the teaching and learning of primary mathematics. To each book the editor, Ian Thompson, has attracted a team of expert contributors, and himself set the characteristically high standard of writing and clarity. It was almost accidental that the publication in 1997 of