Slippery Slopes

by Colin Foster

A common way of beginning a mathematics topic is to announce a definition or a result and then ask pupils to work on tasks or exercises that apply what has been offered. I have often found it more interesting to begin the other way round, with a problem that leads to a *need* for definitions or theorems that will help with its solution. Working this way can be more enjoyable and challenging, but is also more dangerous, because the lesson may unfold in unforeseen ways!

I decided to introduce the concept of gradient to my year 8 class by means of the drawing in Figure 1.

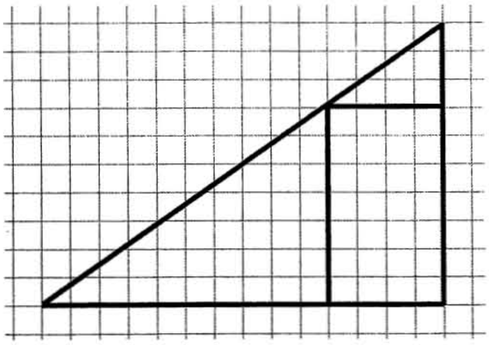

Fig. 1

I asked pupils to calculate some different areas. I didn't state a lesson objective (in this case that would have killed off the lesson!); as far as the pupils were concerned, we were reviewing area calculations. I reassured those who thought that this was too easy – "Trust me; you'll find something interesting!" I knew what I was expecting. The area of the large triangle (the whole shape) comes to $\frac{1}{2} \times 14 \times 10 = 70$ square units. The total area of the two smaller triangles and the rectangle comes to $\frac{1}{2} \times 10 \times 7 + \frac{1}{2} \times 4 \times 3 + 4 \times 7 = 69$ square units.

Everyone noticed the contradiction and most assumed that they had made a mistake, often checking products like 4×7 on a calculator! There was a lot of energy resulting from the need to resolve these conflicting answers. At this stage I mainly listened and encouraged pupils to think out loud. Explanations came quickly: many felt the problem was that the sloping line did not pass exactly through grid points (i.e. it was not a 45° line). This made me worry about their understanding of the area of a triangle as for *any* triangle. With hindsight, I don't think they necessarily had doubts about the calculation of areas of non-isosceles-triangles: sometimes, any explanation seems better than none, and there was a slightly panicky atmosphere, in which pupils wanted to grasp *some* resolution quickly, even if it wasn't completely convincing. The difference of just 1 square unit led some to suggest 'rounding errors', which is not far off the truth, but didn't see where non-integer answers could come from.

Many pupils were willing to work with the problem in their own way, discussing possible answers with one another. Others were unhappy, but unsure what to do. So I offered two possible ways to proceed:

1. Try making a *really accurate* copy of the drawing on 1 cm squared paper.
2. Look at a similar puzzle (Figure 2) for clues (Wells, 1992). The pieces in the top square have a total area of $8 \times 8 = 64$ square units, whereas the same pieces arranged differently seem to make a rectangle of area $13 \times 5 = 65$ square units. (If you make this one you can cut up the pieces and shuffle them around.)

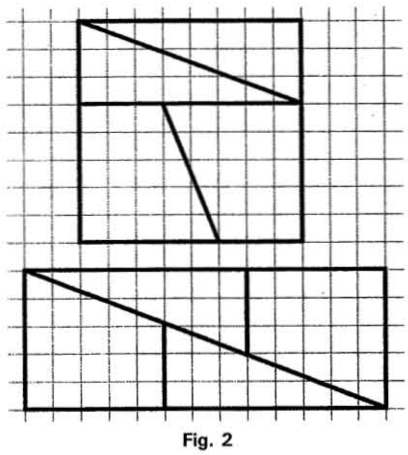

Fig. 2

Sometimes I gave more explicit hints:

Me: Why do you think I made the lines so thick?
Pupil: So they stand out?
Me: Partly, but I'm afraid there's a more sinister reason!

Pupils with better drawing skills (and sharper pencils) began to realize that the thick lines were concealing the fact that the two small triangles in Figure 1 had hypotenuses of different slopes ($\frac{7}{10}$ = 0.7 and $\frac{3}{4}$ = 0.75), so that a greatly exaggerated drawing might look like Figure 3. The obtuse-angled shaded triangle contains the missing 1 unit of area. (That this area is 1 unit of area actually follows from Pick's theorem, since the vertices lie on grid points and there are none inside or along the edges.)

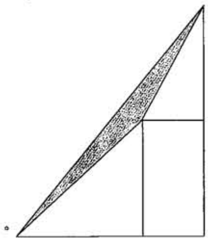

Fig. 3

As pupils tried to explain this (of course, without calculating slopes or using words like 'gradient'), I could introduce the notion of gradient very naturally. A feeling that '10 along and 7 up isn't the same as 4 along and 3 up' could be formalized, and the definition seemed easy and helpful.

These puzzles work when gradients are close but not equal, arising from nearly but not exactly equivalent fractions. (The paradox in Figure 2 relies on the fact that $\frac{3}{8}$ is less than $\frac{2}{5}$, but only slightly.)

I would have liked to look at the coordinates of collinear points, and ask pupils to come up with a method for deciding without drawing whether three points lie in a straight line or not, but there was insufficient time. I encouraged pupils to experiment with pairs of nearly equal fractions like these to try to construct their own puzzles, but it isn't easy! I did feel, though, that the concept of gradient was more firmly understood than when I have taught it by simply stating that "there's this thing called gradient, and this is what it is, and this is how you work it out." Unfortunately, the tendency or requirement to state learning objectives precisely at the beginning of a lesson can make it difficult for concepts to emerge naturally in a problem-solving setting like this.

Reference

This well-known "Chessboard Paradox" goes back to William Hooper (1774) *Rational Recreations*, cited in Wells, D. 1992 *The Penguin Book of Curious and Interesting Puzzles*, Penguin, p. 41. For an animated version, see **http://www.cut-the-knot.org/Curriculum/Fallacies/FibonacciCheat.shtml. See also Curry's Paradox, http://www.mathematik.uni-bielefeld.de/~sillke/ PUZZLES/jigsaw-paradox.html**

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