Questions Pupils Ask!

Unequal Reasoning

by **Colin Foster**

When solving equations, the mantra *whatever you do to one side you do to the other* is widely used to encourage a balancing approach. Of course, 'whatever' does not mean absolutely anything you can think of. For example, 'doubling the first term' of each side would not preserve equality:

$3 + 5 = 1 + 7$
$6 + 5 \neq 2 + 7$.

Pupils would say that you have to double *all* of the terms to preserve the equality:

$2(3 + 5) = 2(1 + 7)$
$6 + 10 = 2 + 14$.

However, when there is a mixture of multiplication and addition, pupils can easily get confused:

$3 + 5 = 2 \times 4$
$2(3 + 5) = 2(2 \times 4)$
$6 + 10 = (2 \times 2) \times 4$
or $2 \times (2 \times 4)$
but not $(2 \times 2) \times (2 \times 4)$.

After grappling with these sorts of issues with my Year 9 class recently, we moved on to some work on inequalities, and I suggested that a balancing approach would work in the same sort of way as with equations:

Me: If the scales are unbalanced with, say, the right-hand side down and the left-hand side up, then if you add or subtract the same amount to both sides, they will *still* be unbalanced in the same direction and by the same amount (Note 1).

As my Year 9 class worked, Esther (Note 2) asked a question:

Esther: Why does it matter what you do to an inequality? If it's not equal, then you can do more or less anything you like to both sides and it *still* won't be equal. You could add 4 to one side and times the other side by 10.

This question made a lot of sense to me. Why were we being so careful with these inequalities, tiptoeing round so as not to disturb the scales when they were unbalanced anyway?

If $x + 3 < 10$ then I could take away 3 from both sides to get $x < 7$, but I could just as well take away 3 from the left-hand side and take away 1 from the right-hand side, or not bother to take anything at all away from the right-hand side. If the right-hand side was bigger than the left-hand side before, it would certainly still be bigger now (even more so!), so couldn't I just write $x < 10$?

In fact, I could add as much as I wanted to the right-hand side, so $x < 1\,000\,000\,000\,000\,000$ would still be true. Preserving *equality* is hard, but preserving *in*equality is easy! It is as though Esther felt that she was trying to solve the 'inequality' $x + 3 \neq 10$.

So long as you avoid x being 7, you can write anything you like as the next line:

$$\frac{42x^3}{65} + \sqrt[5]{x} - 2 = 5x + 1 \text{ or } x = \pi.$$

When working on equations, I'm sure that I had said repeatedly:

It's very important to do the same thing to both sides of an equation because otherwise they won't still be equal.

But can you complete this sentence?

It's very important to do the same thing to both sides of an inequality because otherwise...

The problem here seems to be what we mean by writing a line of algebra *underneath* another one. Progress down the page corresponds to what exactly? It is not merely a random list of true statements: they are connected. Pupils sometimes write $\Rightarrow$ signs down the left-hand side of the page, but when questioned about this they have often given me the impression that it means little more to them than 'and then I did ...'. Yet interpreting each line as *implying* the next doesn't resolve the problem. You can certainly say that

$x + 3 < 10$
$\Rightarrow x < 10,$

because every x that satisfies the first inequality satisfies the second. But the reverse is clearly not true:

$x + 3 < 10 \nLeftarrow x < 10.$

(There are numbers, such as 8, that satisfy the right-hand inequality but not the left.) So progressing from $x + 3 < 10$ to $x < 10$ is 'losing information'. To be useful, the subsequent inequality must not only contain all of the possible values – it must *not* contain *any other values* that *don't* satisfy the starting inequality. This seems like an important but potentially tricky point.

For me, this relates to the distinction between 'finding solutions' to equations (e.g. by trial and improvement or inspection) and 'solving' equations, where in the latter case we find *all the possible solutions* that there can be – and know that we have done so. In order to *solve* an equation or an inequality, movement from one line to the next must be *equivalence* ($\Leftrightarrow$), so that it would make equally good mathematical sense if you were reading *up* the page from bottom to top. This is not too much of a problem with linear equations, but it can be with quadratics. For example:

$x = 4$
$x^2 = 16$
$x = \pm 4.$

Here, in two apparently legitimate steps ('doing the same things to both sides'), we have created a spurious solution ($x = -4$) out of nowhere because the second step is $\Leftrightarrow$ but the first is only $\Rightarrow$.

Pupils tend to meet solving inequalities around Year 9, before they have much/any experience of quadratics (Note 3). I am beginning to think that this may be a good opportunity for thinking about 'if and only if' ('iff'). I have tried using 'iff' with Year 7 classes and have found that pupils can understand and use statements about polygons such as:

A quadrilateral is a rhombus ***if*** *it is a square.*

A quadrilateral is a rhombus ***iff*** *its sides are all equal.*

When older pupils meet quadratic inequalities, absurdities such as:

$x^2 < 16$
$x < \pm 4$

are common. One way to tackle these is to use a number line or a sketch of the graph $y = x^2$, and although this may help them to see that the solution set is in fact $-4 < x < 4$, without some discussion of what is happening to both sides of the inequality, it may leave them unsure why their 'do the same thing to both sides' procedure has failed them. Why does square-rooting both sides not seem to work for inequalities?

I decided to offer my Year 9 class some simultaneous inequalities such as the following, in which there was more than one integer 'answer':

Myron is thinking of a number.

When he multiplies it by 4 and adds 10 he gets more than 185.

When he takes his number away from 100 he gets more than 54.

What might his number be?

Sometimes such scenarios lead to just one integer possibility, which risks perpetuating the idea that 'proper' mathematics questions have just one right answer. I have often found pupils to be uneasy about solving inequalities because they don't feel that they are getting 'a proper answer' ("We still don't know what x is!"). In some styles of mathematics examination, underneath the space for the solution there is a short dotted line on the right-hand side of the page for 'the answer'. So if pupils are solving an equation such as $3x - 4 = x + 6$, they write down their steps and then they put '$x = 5$' on the little dotted line as their 'answer'. In fact, they often just put '5', and this would be marked correct. But what if instead they are solving the inequality $3x - 4 > x + 6$: what goes on the little dotted line then? Certainly '$x > 5$' would be correct, but just a '5' on its own would surely be completely *wrong* – one thing we do know is that x *cannot* be 5.

I hoped that by trying to work out what numbers 'Myron' could have been thinking about, pupils would be keen not to lose any possible numbers at any stage but also not to include any wrong ones. If you are trying to find Myron's number, it is no good to move from saying that $4m + 10 > 185$ to saying that m could be larger than 1000. It could be, but we need to retain at each stage *every possible m* that Myron might be thinking of, in case that happens to be his number, but also *not* to introduce any others that he *couldn't* have been thinking of. We worked in this way, with pupils thinking up 'secret numbers', represented by the initial letters of their names, and then offering inequality clues to enable others to determine what their number might have been.

I left this topic feeling that there is much more to inequalities than merely 'do the same as with equations but put an inequality sign instead of an equals sign in the middle'! In particular, older pupils meet *double* inequalities. If $5 < x + 3 < 10$, then why exactly does it follow that $2 < x < 7$?

Do we envisage a complicated set of scales with *three* pans, with a pan on the left lower than a central one, which is lower than the pan on the right? Is it obvious to pupils that taking 3 away from each pan preserves the ordering?

I could be accused of 'over-thinking' this, but I am always uneasy about topics such as this one where procedural competence can easily outstrip understanding. I don't want pupils to be happily manipulating symbols (even if they are getting the right answers) if they don't have much mathematical awareness of what they are doing and why.

Notes

1. I wondered afterwards what exactly I meant by 'the same amount'. When I mentioned this discussion to my sixth-form class, who had studied moments in mechanics, they pointed out that the resultant moment is proportional to the *difference* between the weights in the two pans, so quantities *added* to both pans do 'cancel out'. But when *multiplying* both sides of an inequality by a factor, the resultant moment will become that many times as large. However, if you think about the resulting angular acceleration of the balance, this should remain the same, since the addition of the weights will increase the moment of inertia too by exactly the same factor.
2. A pseudonym.
3. It may be unfortunate that the first 'quadratic equations' that pupils tend to solve are ones where the answer is known to be a (positive) length: the last step in a Pythagoras' theorem question or calculating the radius of a circle given its area. They may pick up the idea from this that $x^2 = 16$ means that $x = 4$, without realizing that a negative solution has been deliberately discarded.

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