Questions Pupils Ask!

Why isn't π a whole number?

by **Colin Foster**

π holds a magical fascination for pupils and teachers alike. For many years I had the first 1000 or so digits winding around my classroom walls, and it was frequently a talking point. Occasionally, Year 7 pupils would bring in their friends from other classes at lunchtime to show them. Pupils were often interested in how many digits they or I could remember and I was frequently asked whether the digits displayed were the 'actual' ones or just random numbers to illustrate the point that the digits go on forever (Note 1). As a joke, some sixth-form students once swapped around two 10-digit strips to see if I would notice ('Ah, I just knew something felt wrong when I came in today!'). Several times, pupils commented on the fact that the digits made a ring around the room that was reminiscent of a circle, and also that the never-ending path around a circle corresponded to the endless decimal expansion. Many interesting questions arose, such as 'How did you work them out?', 'Are the digits random? How can they be if they never change?' (Note 2) and 'How come you get six 9s in a row there?' (Note 3).

π competes with $\sqrt{2}$ for being the first irrational number pupils are likely to meet in school. Some time around Year 9, pupils encounter both Pythagoras's theorem, leading to surds, and π, and I have sometimes found that they associate these, maybe thinking that 'pi' stands for 'Pythagoras' (rather than that pi is the first letter of his name). (They also often seem to think that pi is connected with pie charts – I suppose because they are circular.) The fact that, unlike the surds, π is transcendental is probably not something that pupils appreciate at this stage. Nor is the fact that these 'weird' numbers really form the majority of the number line, and that the rational numbers that we mostly deal with in school are, in comparison, rare.

I think that the way in which π is often introduced can be puzzling for pupils. They are told that π is 'the ratio of the circumference to the diameter of a circle', and they perhaps measure these values for several cylinders and divide them to obtain approximations for π. But then they are suddenly told that π is *not* a ratio after all! The irrational nature of π is usually defined by saying that it can't be expressed as 'one integer divided by another', but this can be quite confusing, because pupils may think that if π is not the ratio of two *integers* then it must be the ratio of two *decimals*. (They probably divided two decimal measurements to obtain their value for π when measuring their circular objects.) This is an example of where mathematicians' efficient language can be misleading. It is not just that π can't be expressed as the ratio of two *integers*; it can't be expressed as the ratio of any two *rational numbers*. Any ratio of rational numbers, such as

$$\frac{a}{b} \div \frac{c}{d}$$

(where a, b, c and d are positive integers), can be simplified to a ratio of integers – in this case

$$\frac{ad}{bc}.$$

So saying 'a ratio of integers' is really just a more elegant way of saying 'a ratio of rationals'. The business about integers or rationals is just to stop us writing things like

$$\frac{\pi}{1} \text{ or } \frac{5\pi}{5}$$

If you want something to equal π, it seems as though you have to put the π into it. However, you can express π as an *infinite* sum of rational numbers, such as

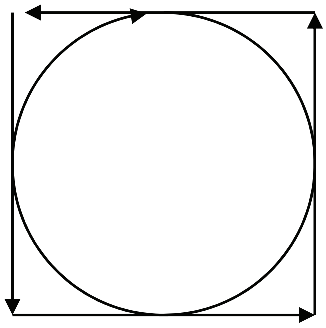
$$\pi^2 = 6\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\right)$$

just as you can write it in its usual form as an infinite non-terminating and non-recurring decimal. It is not known whether there might be some exact way to express π in terms of other transcendental numbers, such as e (Note 5). The fact that π is irrational means that, for any circle, either the diameter or the circumference (or both) must be irrational.

So why isn't π a whole number? You might be inclined to reply 'Why should it be?' A colleague of mine suggested asking 'Which whole number would you like it to be?' But there does seem to be a feeling that a circle is a very neat and symmetrical and beautiful thing, and why should it have this weird, awkward number associated with it? In 1897, the Indiana legislature considered a bill involving several different implied values for π, but it seems that the idea that they passed a law legislating a value for π is, sadly, a myth (Dudley, 1999). Perhaps π is just a brute fact: it is what it is, no matter what anyone says, and simply can't be explained? On the other hand, if π *were* exactly three, all sorts of exciting things would happen (Griffiths, 2006)!

The analogy of running can help learners to obtain intervals in which π must lie.

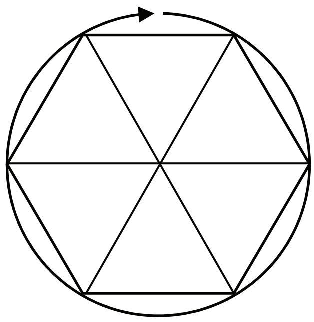
Which is faster – to run around the square or to run around the circle?



Pupils can get distracted by worrying about turning at the corners, but the question is really intended to focus on the total distance run. Clearly the circle is shorter because it 'cuts the corners'. So, since the sides of the square are equal to the diameter d of the circle, we have circumference $c < 4d$. It is nice just to stare at the diagram and 'see' this.

Now we try a regular hexagon instead.

Which is faster – to run around the perimeter of the hexagon or to run around the circle?



This time the hexagon perimeter is shorter, because it links points on the circle by *straight* paths rather than curvy ones. Since two sides of one of the equilateral triangles make a diameter, the perimeter of the hexagon is $3d$, so now we can see that $c > 3d$.

So, purely visually, $3d < c < 4d$, meaning that π can't be an integer – it must be some number between 3 and 4. It can't be 3 because a circle is longer than a hexagon. Of course, this doesn't prove that π is irrational, but it does show that it is non-integer, which is a start (Note 4). I like this demonstration but, although it convinces, it doesn't, for me, really explain *why* in any kind of satisfying or deep way. I have sometimes taken a class outside and found a large circular object in the school grounds, wrapped string around its circumference and laid out the length of string repeatedly along the diameter: 'One, two, three – and a bit left over'. It can be a memorable demonstration, but I remember once thinking that the reason for this 'three and a bit' was just as mysterious to me as it was to the pupils. I certainly feel that I haven't come close to finding a good response to the pupil's question – I seem to be saying that 'it just is'. If any readers have better responses, I would be very interested to hear them.

Notes

1. They were, of course, correct – at least I believe so! I obtained the poster pieces from **www.cleavebooks.co.uk/trol/trolgf.pdf**.
2. Rather like the 'random' digits in one of those old-fashioned tables of 'random numbers'.
3. From the 762nd decimal place (see **www.angio.net/pi/**).
4. Proving that π is irrational is much much harder than proving that $\sqrt{2}$ is (which can be understood by a Year 7 pupil). Ivan Niven's (1915–1999) proof of π's irrationality **http://en.wikipedia.org/wiki/Proof_that_%CF%80_is_irrational#Niven.27s_proof** is the 'easiest' one I have come across, and might be just accessible to an upper-sixth student.
5. Euler's identity $e^{i\pi} + 1 = 0$ is a beautiful relationship between π and e, though not an 'algebraic' one. It is not known whether π and e are algebraically independent (see *Schanuel's conjecture*). In a 1975 spoof article in *Scientific American*, Martin Gardner famously claimed that *Ramanujan's constant* $e^{\pi\sqrt{163}}$, an 'almost-integer', was in fact an integer.

References

Dudley, U. 1999 'Legislating Pi', *Math Horizons*, **6**, 3, pp. 10–13.

Griffiths, J. 2006 'Could π be Three?', *Mathematical Gazette*, **90**, 517, pp. 103–107. Available at **www.s253053503.websitehome.co.uk/articles/mydirr/pi-is-3.pdf.**

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