STUDENT-GENERATED QUESTIONS *in Mathematics Teaching*

Colin Foster

Exploring even something as simple as a straight-line graph leads to various mathematical possibilities that students can uncover through their own questions.

Too often the discourse of the mathematics classroom is defined as the teacher asking the questions and the students answering them—or trying to (Dillon 1990; Susskind 1969, 1979). I once observed a teacher begin a mathematics lesson by drawing a diagram on the board. The students watched in silence. As the teacher turned around to face the class, a hand immediately went up. The teacher looked frustrated. "How can you have an answer already?" he asked. "I haven't told you what the question is yet!" The hand went down, and the other students laughed, suggesting that they understood the unwritten rule of this class: The teacher asks the questions, and the students wait their turn to respond. No doubt the teacher had a good question in mind, but perhaps the student did too.

Tizard and colleagues (1983) have lamented the paucity of student-generated questions: "The children seem to learn very quickly that their role at school is to answer, not to ask questions" (p. 279). This observation appears to be particularly true of girls (Pearson and West 1991) and is by no means a recent problem; as early as 1940, Corey was commenting on the same phenomenon. More recently, Dillon (1988) has contrasted the reluctance of students to ask questions in the classroom with the plethora of questions they typically ask in other circumstances. He comments: "Questioning is frequently used in classrooms, but rarely as a knowledge-seeking method. Those who ask questions—teachers, texts, tests—are not seeking knowledge; those who would seek knowledge—students—do not ask questions" (p. 197).

Certainly teachers should not be prohibited from asking questions, but if students are always placed in the position of responding rather than initiating, then we can hardly be surprised if at times they seem passive and flounder when given open-ended tasks. Questioning techniques continue to be a popular topic for mathematics in-service teacher training, and this approach is no doubt helpful.

But students also need to develop their ability to ask mathematical questions. The questions students choose to pose give insight into what they know and what they see as mathematically significant and so can be a

useful indicator to the teacher of students' progress (Maskill and de Jesus 1997). More important, mathematics is inherently an inquisitive discipline. The subject develops because as soon as a mathematician has answered one question, he or she is very likely to think of others. Without questions, there would be no mathematics, and unless students learn to ask questions, they cannot operate as mathematicians.

Teachers frequently use questions to maintain control in the classroom, and a reluctance to allow students to ask questions may betray some insecurity: "If the students start asking whatever questions they like, what if we don't know the answers? What if students end up off topic?" The only solution is to recognize that no one can have all the answers; instead, a willingness to engage thoughtfully with whatever is raised is far more important—and could indeed be said to model an intelligent and mature approach to life generally. A reassuring feature of mathematics is its connectedness: If the teacher does not know the answer to exactly what the student is asking, he or she is still likely to know something related that will enable him or her to support the student as they work on it together. This connectedness also means that straying into another area is often not as much a diversion as it may initially appear to be.

I have found that students seem to get better at asking questions as they have more opportunity to do so. I often start lessons with some mathematical object—a diagram, an equation, a puzzle, or a paradox—that seems to me provocative or rich in some way. Then I simply ask students to look at the object and see what mathematical questions they can ask. Often I do not even need to do that much—the lesson begins when the first student speaks. The person asking the question does not risk being put on the spot to answer it. And I certainly do not feel that it is my responsibility as the teacher to try to answer any of the questions myself. Sometimes I may know an answer (or something relevant), and sometimes I do not. I generally collect the questions offered by writing them on the board. If I get an occasional "irrelevant" question, it is not the end of the world; such a question may indeed turn out to be more useful than it initially seems, or it can always be modified or discarded later.

QUESTIONS FROM A GRAPH

Following is an example of a possible starting point, together with some questions generated by students.

> Sketch the graph $3x + 4y = 24$ (see **fig. 1**). What mathematical questions can you ask?

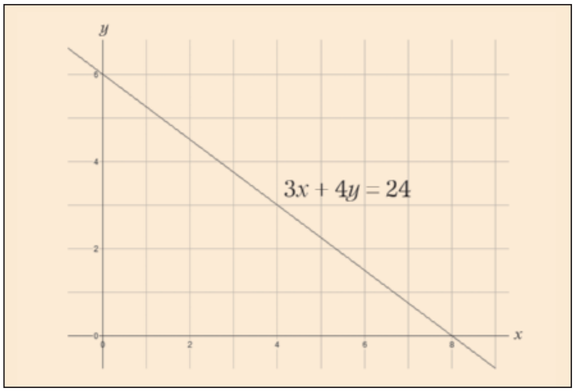


Fig. 1 What mathematical questions can you ask about the graph of $3x + 4y = 24$?

1. What Angles Are Formed between the Line and the x- and y-Axes?

The angles produced are $\tan^{-1}(3/4)$, $\pi - \tan^{-1}(3/4)$, $\tan^{-1}(4/3)$, and $\pi - \tan^{-1}(4/3)$, if we count only the positive angles whose magnitude is less than π. Students might think about why, with a general line, having more than four different angles is impossible and might explore lines that create fewer angles with the axes.

Some examples would be the family $x \pm y = k$ where k is any nonzero constant, which gives only two angles ($\pi/4$ and $3\pi/4$); $y = \pm x$, which gives only one angle ($\pi/4$); and $x = k$ and $y = k$ where k is any nonzero constant, which each gives one size of angle ($\pi/2$). The line $y = kx$ where k is any constant other than 0, 1, or –1 also gives two angles: $\tan^{-1}|k|$ and $\pi - \tan^{-1}|k|$. Some good, exhaustive logical thinking is required to dispose of all the possible cases. While exploring, students may notice (and perhaps prove) results such as

$$\tan^{-1} a + \tan^{-1}\left(\frac{1}{a}\right) = \frac{\pi}{2}$$

and

$$\tan\left(\frac{\pi}{2} - a\right) = \frac{1}{\tan a} \quad (a > 0).$$

2. What Are the Equations of Another Three Lines That, Together with the Segment of This Line in the First Quadrant, Make a Square?

Students might not immediately appreciate that there are two possible squares, one extending upward and one downward from the given line. In each case, the two parallel lines $3y = 4x + 18$ and $3y = 4x - 32$ are needed, but the third line can be either $3x + 4y = 74$ (for the square extending upward from the given line) or $3x + 4y = -26$ (for the one extending downward). Students could try starting with a different line and looking at how this change affects the equations of the other three

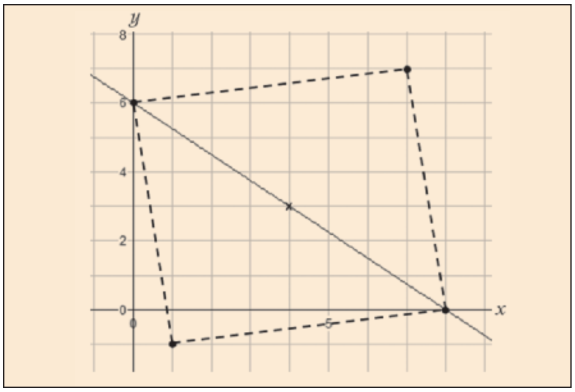


Fig. 2 A square can be constructed given the diagonal.

lines needed to make a square. They could also try making other regular polygons, such as an equilateral triangle, so that the given line segment forms one of the sides.

Another question would be to ask for the vertices of the square that has the line segment from (0, 6) to (8, 0) as its *diagonal* (Hedger and Kent 1978). A common first response is (0, 0) and (8, 6), but of course these vertices form a rectangle, not a square. The other two vertices are in fact (7, 7) and (1, –1) (see **fig. 2**), and we could also ask for the equation of the line containing the other diagonal of the square, which is $3y = 4x - 7$.

3. What Are the Area and the Perimeter of the Triangle Formed by the Line and the x- and y-Axes?

The area is readily calculated as (6 • 8)/2 = 24 square units, and because we have a Pythagorean 6-8-10 triangle, the perimeter is also found to be 24 units—hence, this is an *equable triangle* (a triangle whose area is numerically equal to its perimeter). This result is likely to provoke questions: Why did this happen? Is the result unique? I deliberately chose a starting line that gave a triangle with this property in case students looked at area and perimeter. We teachers should think carefully about the specifics of any initial example we provide, because this choice can dramatically affect possibilities later on.

For the general line $ax + by = c$ (with $a, b, c > 0$), the coordinates of intersection with the x- and y-axes can be found by substituting $y = 0$ and $x = 0$, respectively, and solving. This gives $ax = c$ and $by = c$, so $x = c/a$, and $y = c/b$. Therefore, the x-intercept is $(c/a, 0)$, and the y-intercept is $(0, c/b)$. The area will then be

$$\frac{1}{2}\bullet\frac{c}{a}\bullet\frac{c}{b}=\frac{c^2}{2ab}.$$

We calculate the length of the hypotenuse:

$$\begin{aligned}\sqrt{\left(\frac{c}{a}\right)^2+\left(\frac{c}{b}\right)^2} &= \sqrt{\frac{b^2c^2+a^2c^2}{a^2b^2}}\\ &= \sqrt{\frac{c^2\left(b^2+a^2\right)}{a^2b^2}}\\ &= \frac{c}{ab}\sqrt{a^2+b^2}\end{aligned}$$

Thus, the perimeter is

$$\frac{c}{ab}\sqrt{a^2+b^2}+\frac{c}{a}+\frac{c}{b}=\frac{c}{ab}\left(\sqrt{a^2+b^2}+b+a\right).$$

Setting the expressions for the area and the perimeter equal, we obtain

$$\frac{c^2}{2ab}=\frac{c}{ab}\left(\sqrt{a^2+b^2}+a+b\right).$$

Because none of a, b, or c is zero, we have

$$c=2\left(\sqrt{a^2+b^2}+a+b\right),$$

which will be an integer if and only if $\sqrt{a^2+b^2}$ is an integer. This will happen if and only if a and b are the legs of a Pythagorean triangle because then $\sqrt{a^2+b^2}$ will be the length of the hypotenuse.

For example, choosing $a = 3$ and $b = 4$ gives $c = 24$ and our initital equation. This equation will be obtained with any multiple, such as $a = 6$ and $b = 8$, giving $c = 48$ and the equation $6x + 8y = 48$, which is equivalent to $3x + 4y = 24$. A different primitive Pythagorean triple produces a different equation, such as $a = 5$ and $b = 12$, giving $c = 60$ and the equation $5x + 12y = 60$, leading to a triangle with area and perimeter 30. These are the only two equable right triangles with integral sides.

4. How Many Possible Routes Are There from (0, 6) to (8, 0) If Integer Steps Right and Down Are the Only Ones Allowed?

Estimate an answer before calculating and check the estimate later. This problem may remind students of Pascal's triangle, the number of routes to any lattice point ($x \ge 0$, $y \le 6$) being the total of the number of routes to the points 1 to the left of that point and 1 above that point because we must have come through one of these points. For example, the number of routes from (0, 6) to (3, 4) is the total of the number of routes from (0, 6) to (2, 4) and the number of routes from (0, 6) to (3, 5). Labeling each lattice point with the number of routes from (0, 6) to that point, we see that the number of routes to (x, y) is ${}_{6+x-y}C_x$, so in particular the number of routes from (0, 6) to (8, 0) will be ${}_{6+8-0}C_8 = {}_{14}C_8 = 3003$, probably

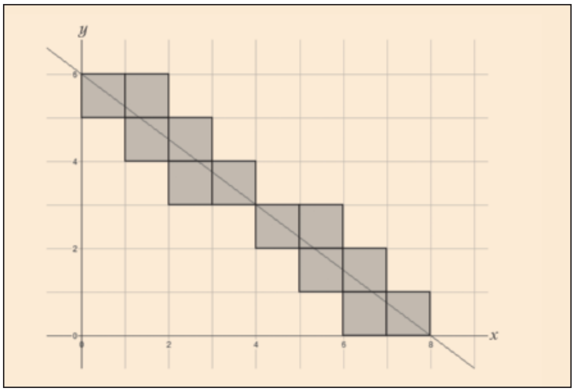


Fig. 3 The line $3x + 4y = 24$ passes through 12 squares in the first quadrant.

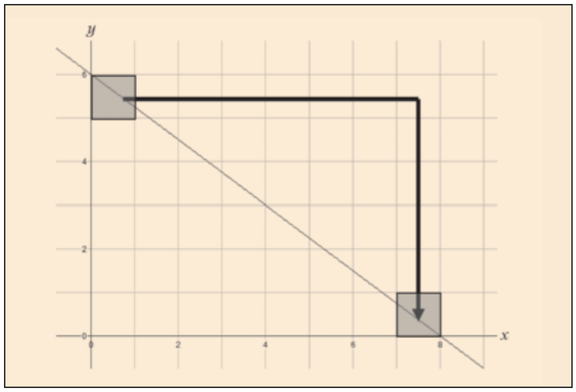


Fig. 4 Moving horizontally and then vertically takes us through 13 squares.

considerably higher than most students would have estimated at the beginning.

5. How Many Squares Does the Line Pass through in the First Quadrant?

The given line passes through 12 squares (see **fig. 3**). To move from (0, 6) down to (8, 0), it seems that we must pass through at least 13 squares (see **fig. 4**). This number comes from the 8 squares traversed horizontally plus the 6 squares traversed vertically; we must remember to subtract the square at the top-right corner where we make our turn, which otherwise we would count twice. So, in general, we might expect the number of squares that the line passes through to be $x + y - 1$, where x and y are the x- and y-intercepts of our graph, respectively (provided that x and y are both positive integers).

But this hypothesis does not take into account the possibility of a line passing *exactly through* a lattice point. Every time the line hits a lattice point, we will lose a square from our total. The line will pass through lattice points every time x and y have a common factor. Thus, the number of squares equals $x + y - GCF(x, y)$, where GCF indicates finding the greatest common factor of the two values. Here, $GCF(8, 6) = 2$, so the number of squares the line passes through is $8 + 6 - 2 = 12$. For the general line $ax + by = c$, we saw above that the x- and y-intercepts are $x = c/a$ and $y = c/b$. Assuming that these intercepts are integers, we obtain the number of squares: $c/a + c/b - GCF(c/a, c/b)$.

6. How Many Whole Squares Are Contained in the Region Enclosed by the Line and the Axes?

As **figure 5** shows, the given line and the axes enclose 18 whole squares. Clearly, for the general line $ax + by = c$, the expression $c^2/(2ab)$ $(a, b \neq 0)$ derived above for the area (in this case, 24) will be an upper bound. However, we can do better than that. The line must pass through at least $\max(x, y)$ squares, where $\max(x, y)$ is equal to whichever of x and y is bigger and where x and y are the x- and y-intercepts, respectively, assuming again that these are integers. As before, because the x- and y-intercepts are $x = c/a$ and $y = c/b$, respectively, we can reduce our upper bound to

$$\frac{c^2}{2ab} - \frac{1}{2}\max\left(\frac{c}{b}, \frac{c}{a}\right),$$

assuming again that a and b both divide c.

In the case of $3x + 4y = 24$, the computation brings our upper bound down to $24 - 4 = 20$. But when we think about why we are still 2 over the correct number—because the line passes through exactly one lattice point (4, 3)—we have a means of solving the problem precisely. By exploiting symmetry, we can superimpose **figure 3** on **figure 5** and obtain the diagram shown in **figure 6**. There the dotted squares above the line are an exactly congruent copy of the dark gray squares below the line, merely rotated half a turn about (4, 3), the center of the line segment joining (0, 6) to (8, 0).

So if n is the number of squares below the line, then it follows by adding areas that $2n$ + (the number of squares the line passes through) equals the area of the whole rectangle. Assuming again that both a and b divide c, we can write:

$$2n + \frac{c}{a} + \frac{c}{b} - GCF\left(\frac{c}{a}, \frac{c}{b}\right) = \frac{c}{a} \bullet \frac{c}{b}$$

From this, we have the following:

$$2n + \frac{bc + ac}{ab} - GCF\left(\frac{c}{a}, \frac{c}{b}\right) = \frac{c^2}{ab}$$

$$2n = \frac{c(c - a - b)}{ab} + GCF\left(\frac{c}{a}, \frac{c}{b}\right)$$

$$n = \frac{c(c - a - b)}{2ab} + \frac{1}{2}GCF\left(\frac{c}{a}, \frac{c}{b}\right)$$

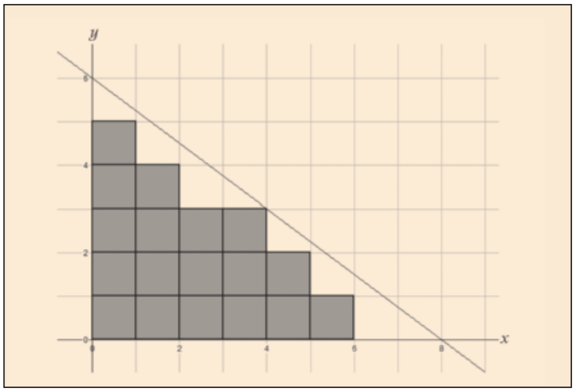


Fig. 5 There are 18 whole squares underneath the line $3x + 4y = 24$.

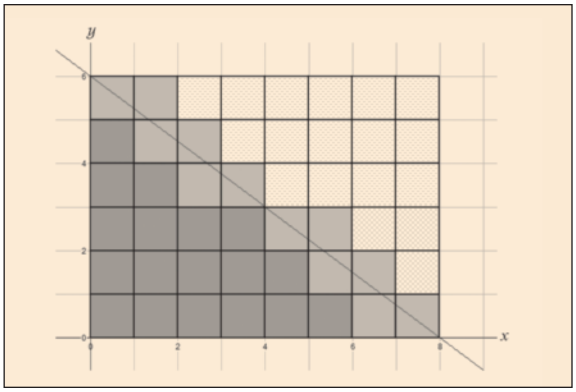


Fig. 6 We can use symmetry to compute the number of whole squares enclosed by the line and the axes.

CONCLUSION

If questions are slow in coming, be patient—this approach may be a big departure from students' expectations of what constitutes a mathematics lesson. Once the class has generated several possible questions, students could vote on which one or two to work on first, or different students (perhaps in groups or pairs) could choose a question to try, sharing their progress later.

A questioning atmosphere in the mathematics classroom goes far beyond the usual clarifying questions, in which students ask teachers to repeat or re-explain points that they have made. I remember university lecturers who at the end of a lecture would ask, "Any questions?" and, if there were none, would say "Good!" (obviously taking it as a compliment that they had explained everything thoroughly and clearly). In my classroom, I want to move away from the culture in which students' questions are an interruption to the main event, a necessary evil to be tolerated and dealt with swiftly. Student-generated questions can be the main event, allowing lessons to develop from students' curiosity and building on their developing interest.

In other school subjects, having students answer their own questions is not so easy. Students cannot travel to Uganda during today's geography lesson; the science teacher cannot allow students to try mixing concentrated sulfuric acid and sodium to see what happens. But in the abstract world of mathematics, such obstacles are not generally present. Students can experiment without burning themselves and safely pursue whatever lines they desire, often without needing any particular equipment or teacher direction.

We teachers should take full advantage of this aspect of mathematics by not constantly requiring students to dance to our agenda. Let's give them space to pursue their own mathematical inquiries and develop initiative, interest, and mathematical capability. Perhaps one way to judge the quality of our teaching is to ask whether our students are becoming increasingly adept at asking genuinely mathematical questions about their work.

REFERENCES

Corey, Stephen M. 1940. "The Teachers Out-Talk the Pupils." *The School Review* 48 (10): 745–52.

Dillon, James T. 1988. "The Remedial Status of Student Questioning." *Journal of Curriculum Studies* 20 (3): 197–210.

———. 1990. *The Practice of Questioning.* New York: Routledge.

Hedger, Keith, and David Kent. 1978. "Given Two Points." *Mathematics Teaching* 84: 19–21.

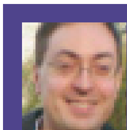
Maskill, Roger, and Helena P. de Jesus. 1997. "Pupils' Questions, Alternative Frameworks, and the Design of Science Teaching." *International Journal of Science Education* 19 (7): 781–99.

Pearson, Judy C., and Richard West. 1991. "An Initial Investigation of the Effects of Gender on Student Questions in the Classroom: Developing a Descriptive Base." *Communication Education* 40 (1): 22–32.

Susskind, Edwin. 1969. "The Role of Question-Asking in the Elementary School Classroom." In *The Psycho-Educational Clinic*, edited by F. Kaplan and S. B. Sarason. New Haven, CT: Yale University Press.

———. 1979. "Encouraging Teachers to Encourage Children's Curiosity: A Pivotal Competence." *Journal of Clinical Child Psychology* 8: 101–6.

Tizard, Barbara, Martin Hughes, Helen Carmichael, and Gill Pinkerton. 1983. "Children's Questions and Adults' Answers." *Journal of Child Psychology and Psychiatry* 24: 269–81.



COLIN FOSTER, c@foster77.co.uk, teaches secondary mathematics at King Henry VIII School in Coventry, United Kingdom, and writes books for mathematics teachers.