Always up to?

by **Colin Foster**

Inequalities can be tricky things to make sense of, especially when they are presented in words. I was puzzled recently by a sign I saw outside a shop:

What does it mean? Having 'up to' and 'less' in the same phrase makes interpretation tricky. The larger font is used for the '60% less' part, so that would seem to be the thing that the shop wants its (potential) customers to focus on (Note 1). But 'always up to' is equivalent to 'never more than', which is hardly much of a boast! Understood this way, it means that its discounts are never better than 60%. Why would you pay for a sign to say that?

On the other hand, perhaps the shop wants 'up to' to be interpreted to mean 'as good as'. This seems unlikely, since many of the items in the shop are discounted less than the headline 60%. Presumably, they do not wish to claim that everything has *at least* a 60% discount – but by avoiding this they then end up saying that *nothing* has *more* of a discount than 60%.

I get similarly confused by the slogan on the local buses which says 'Up to every 15 minutes'. What does 'up to' mean here? Do they mean 'up to 15 minutes', so, for example, at busy times the buses might come every 10 minutes? That would mean that the *longest time* you should expect to wait is 15 minutes, which would be nice. But, on the other hand, perhaps they mean *up to this frequency*, of 1 bus per 15 minutes, meaning that it would include the possibility of 1 bus per *hour*? Understood in this way, all the slogan means is that buses will never come *more* frequently than every 15 minutes – you will never get three coming along in quick succession, for instance, which seems an odd thing to claim! Perhaps the company wants to have this wriggle room so as to give the impression of promising much while actually being able to claim, if challenged, that it is promising nothing? This would be like the mythical job reference for a lacklustre candidate, which said "I cannot say enough good things about this candidate or recommend them too highly" – sometimes ambiguity has its uses!

This got me thinking about the difficulties of capturing mathematical inequalities in words. You cannot underestimate the importance of getting inequalities right. Actually, this is wrong, and the whole point is that you very easily *can*. What I should say is the opposite – that you cannot *over*estimate the importance of getting inequalities right, but people don't always say it that way. I could correctly say that you *should not* underestimate the importance of getting inequalities right, but if I say that you *cannot* underestimate it then I am saying that you can never get lower than it, which means that it must be zero! A *Google* search just now (November 2016) on the phrase 'it is impossible to underestimate the importance' gives about 108 000 results. On a quick look at some of these, they all seem to mean the opposite of what they say! A search for 'it is impossible to overestimate the importance' gives 95 000 results, suggesting that these two apparently opposite phrases may be used about equally often to mean the same thing.

No wonder students find 'least upper bound' and 'greatest lower bound' confusing. At school level, questions often avoid these terms by asking, for example, "What is the lower bound for 2 cm (correct to the nearest integer)?" The question means the *greatest* lower bound, otherwise you could answer zero, but the argument is that, if you included the word 'greatest' in the question, the student might be led to think that an *upper* bound was required. I do wonder, though, whether this supposed simplification itself leads to confusion, as the idea that a measurement has 'a' lower bound can itself be difficult. The language of smallest and largest is much easier than upper and lower bounds, but is also problematic. "A nail measures 2 cm (correct to the nearest integer). What is the smallest length it could be?" is fine, and the answer is 1.5 cm. But "What is the largest length that it could be?" doesn't work, because it can't *actually* be 2.5 cm, as that would round up to 3 cm. This generally leads to students wanting to write 2.49 cm, or 2.499 cm, or even $2.4\dot{9}$ cm – and they are incredulous when told that this is exactly equal to 2.5 cm. (In my experience, students are rarely totally convinced of this equality, however much we point out that $\frac{1}{3}\times 3=1$ and $0.\dot{3}\times 3=0.\dot{9}$ and so on (Note 2)).

Maximum–minimum questions can be even harder to pose unambiguously. For example, suppose you have a lift (an elevator) which can carry a maximum mass of 500 kg and you want to work out the maximum number of people that it can safely carry. You are told to assume that each person has a mass of 80 kg, and it is tempting just to work out $\frac{500}{80}$ and round down to the nearest integer to give 6 people. But of course the catch in these upper/lower bound questions is that you have to worry about the ranges in which these measurements might lie. So maybe you are told that everything is correct to 1 significant figure, so the lift might be able to carry only 450 kg, and each person might be almost 85 kg. That would allow only 5 people to get in. So is that the 'maximum number of people that it can safely carry'?

Well, that is your worst-case-scenario answer for the largest number of people the lift could safely carry if the lift is as weak as it could possibly be and the people are as heavy as they could possibly be. But isn't this therefore the *minimum* number of people, since we have been so cautious? And, since the question asks for the *maximum* number of people, maybe we should make the *opposite* assumptions, and assume that the people are each only 75 kg and that the lift can actually carry up to 550 kg? Making these more favourable (but riskier) assumptions would allow 7 people to travel in the lift (Note 3) – but does this now mean that we're not being 'safe'? It's a bit unclear what we are supposed to be doing here. Are we trying to maximize ($\frac{550}{75}$) or minimize ($\frac{450}{85}$)? Should we focus on 'maximum number' or 'safely carry'? I have heard people argue that it depends on whether you say 'can safely carry' or 'could safely carry', which seems a very fine distinction! Sometimes in answering these questions there appears to be an unwritten 'safety first' rule that prioritizes the minimizing over the maximizing, but in general I think it depends on what exactly you assume. I think that you can make a reasonable case for answering 5 or 7 to this question, depending on how you interpret it, so care really is needed! (Note 4)

Notes

1. Presumably 'less' relates to prices, and not to the quantity or quality of its products!
2. And perhaps that's not such a bad thing if you believe that writing $0.\dot{9}=1$ is more of a decision or axiom than something which we can demonstrate to be true.
3. Note that we still *round down* $\frac{550}{75}$ to 7, even though we are trying to get the maximum!
4. Of course, this is a very suspect real-life scenario, as we would hope that real lifts are operated to a considerably larger margin of error! For a free lesson plan that explores upper and lower bounds in a different context, see Foster (2016).

Reference

Foster, C. 2016 'Hopping Along', *Teach Secondary*, **5**, 4, pp. 31–33. Available free at **http://www.teachwire.net/teaching-resources/hopping-along-time-distance-and-estimation-lesson-plan-for-ks4-maths**

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