HCF and LCM – Beyond Procedures

by **Colin Foster**

In mathematics, the relationship between ‘understanding’ and ‘doing’ is complicated. Rarely do both aspects march along exactly together. At times, ‘understanding’ leads, and practice is perhaps needed, to bring ‘doing’ along to a fluent level. More interestingly, on other occasions ‘doing’ may be ahead of ‘understanding’, and learners may complain, “I’m getting the right answers but I don’t really get it!” This may not seem like a priority for the busy teacher, who may feel that they have bigger problems to deal with: it is tempting to say, “Just carry on and the understanding will follow”, as you rush away to help what seems like a more serious case. But will it? Automatically? Computer algebra systems such as *Wolfram|Alpha* (**www.wolframalpha.com**) are just ‘dumb machines’ but can do many of the processes we expect of learners, without (presumably) anything that we would call ‘understanding’ – and they get far more practice than even the most conscientious learner! If we work with learners on understanding only until the procedure or algorithm ‘clicks’ and then stop, understanding becomes merely a means to a rather dull end. And we leave learners with techniques that they are uncomfortable with and which are not integrated into the rest of their knowledge.

The topic of highest common factors (HCF) (also known as greatest common divisor, GCD) and lowest common multiples (LCM) is frequently approached by learners in quite a procedural manner. It is possible to ‘do it’ accurately and efficiently without much idea of what it means or how it works. Learners are taught to break down numbers into products of powers of primes and then follow various rules (perhaps involving tables or diagrams) to pick out the factors that they need in order to create the HCF or the LCM. Once they have mastered this, the topic is over. But there is much more potential in HCF and LCM than this; in particular, there are some beautiful connections which I am only just realizing myself. I have heard teachers recommend teaching HCF and LCM in different lessons, to prevent learners from muddling them up. But I think it is much more fruitful to examine them together.

For instance, try this problem:

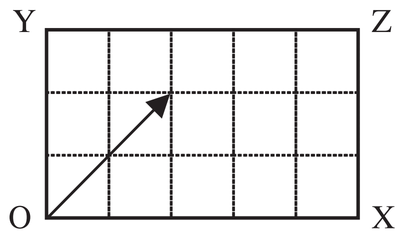
> I’m thinking of two positive integers.
> Their product is 360 and their HCF is 6.
> What is their LCM?
> How many possible answers are there? Why?

You may be surprised that there is only one possible answer and that you can find it without working out what the two numbers are. Do you see why this is?

Here is a problem that may help:

> For which values of a and b is it true that
> $\mathrm{HCF}(a, b)\ \mathrm{LCM}(a, b) = ab$?

In fact, this is true for all positive integers a and b. In a sense, I think I must have known this, but until this week I don’t remember ever seeing it expressed quite so simply – and the other mathematics teachers that I have asked have reacted similarly. It makes the first problem above straightforward and means, in general, that statements in terms of HCF can be replaced by ones in terms of LCM, which is actually what led me to notice it. I had set the ‘Snooker’ task below (Banwell *et al*., 1986) for my Year 8 class and some were expressing their findings in terms of HCF and others in terms of LCM and the two descriptions seemed to be equivalent – and indeed they were.

> **Snooker**
>
> An idealized snooker ball is projected at 45° from one corner of the rectangular snooker table shown below. If it bounces perfectly off the sides and keeps going until it gets to a pocket (O, X, Y or Z), which pocket will it end up in?
>
> 
>
>
> What happens for different shapes and sizes of snooker tables? Why?
>
> The ball always starts out the same way from O.

Thinking about this connection between HCF and LCM, I began to ask other questions:

> Is the HCF of two numbers always a factor of the LCM?
>
> Why/why not?
>
> What about if you have *more than two* numbers?

Here is an extension to that:

> I'm thinking of two positive integers.
>
> What could they be if ...
>
> 1. LCM = HCF
> 2. LCM = 2 × HCF
> 3. LCM = 10 × HCF
> 4. LCM = 12 × HCF
>
> Try this sort of problem with other multipliers.
>
> What do you find?

Specifying a value for the HCF leaves infinitely many possibilities for the two numbers; hence the following kind of task:

> Find two numbers with an HCF of 4.
>
> And another two.
>
> And another two.
>
> Can you work out a method for answering questions like this?
> Can you make both numbers be ... greater than 100?
> greater than 1000?
>
> Why/why not?
>
> Can you make both numbers be in the 6 times table? Why / why not?
>
> Can you make the two numbers add up to 76?
>
> Can you find *three* numbers that have an HCF of 4?
>
> Or *four* numbers?
>
> What happens if you change the 4 to another number?

But specifying a value for the LCM does not, suggesting tasks such as this:

> How many pairs of numbers have an LCM of 12? Why?
>
> How many sets of *three* numbers have an LCM of 12?
>
> And so on…

Venn diagrams help to reveal the connections between HCF and LCM if we write the prime factors in the appropriate regions. Figure 1 shows a Venn diagram for the numbers 12 and 30. The HCF is the *intersection* of the two sets and the LCM is the *union*, noting that we *multiply* the elements of the sets to obtain the values. So HCF(12, 30) = 2 × 3 = 6 and LCM = 2^2 × 3 × 5 = 60.

We can see here that HCF × LCM = 6 × 60 = 360, the same as the product of the numbers, 12 × 30, and we can prove this in general. In set theory, it is true that $A \cup B = A + B - A \cap B$. (This is amenable to 'proof by colouring in'!) This means that $\mathrm{LCM}(a, b) = \dfrac{ab}{\mathrm{HCF}(a, b)}$, where multiplication corresponds to the addition of sets and division corresponds to their subtraction. Equivalently, $\mathrm{HCF}(a, b)\,\mathrm{LCM}(a, b) = ab$.

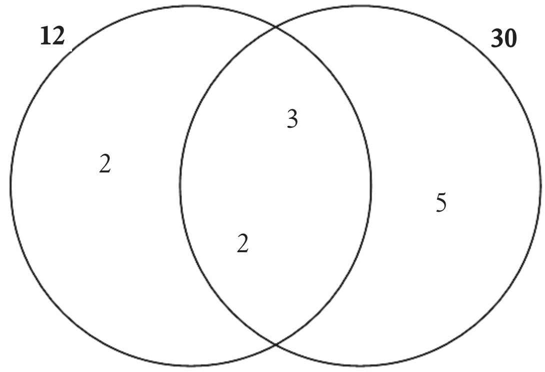


Fig. 1 The numbers 12 and 30

So a Venn diagram approach can be very useful for solving problems such as these:

> If possible, find values of x to satisfy these equations.
>
> If any are impossible, try to say why.
>
> 1. HCF(x, 48) = 6
> 2. HCF(x, 36) = 6
> 3. HCF(x, 12) = 6
> 4. LCM(x, 6) = 48
> 5. LCM(x, 6) = 36
> 6. LCM(x, 6) = 12

Or these:

> Find two numbers with an HCF of 4 and an LCM of 24.
>
> How many possible answers are there? Why?
>
> Find *three* numbers with an HCF of 4 and an LCM of 24.
>
> How many possible answers are there? Why?
>
> Try this sort of problem with other numbers.

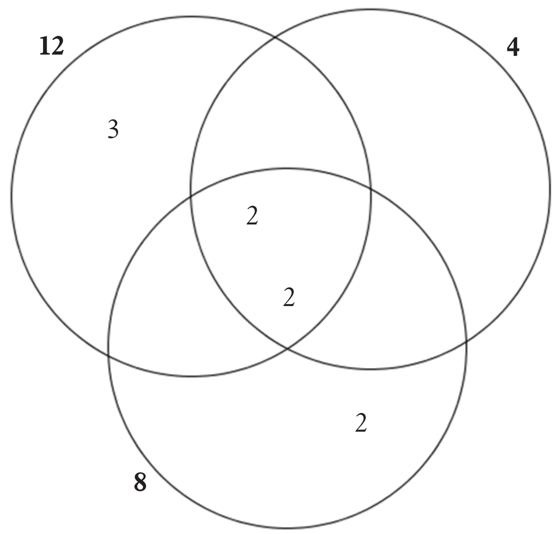


Fig. 2 The numbers 4, 8 and 12

With three numbers (e.g. Fig. 2), we need the relation:

$$A \cup B \cup C = A + B + C - A \cap B - B \cap C - A \cap C + A \cap B \cap C$$

In a similar way to before,

$$\mathrm{LCM}(a, b, c) = \frac{abc\,\mathrm{HCF}(a, b, c)}{\mathrm{HCF}(a, b)\,\mathrm{HCF}(b, c)\,\mathrm{HCF}(a, c)}$$

And so on – with four numbers we use the relation:

$$A \cup B \cup C \cup D = A + B + C + D - A \cap B - A \cap C - A \cap D - B \cap C - B \cap D - C \cap D + A \cap B \cap C + A \cap B \cap D + A \cap C \cap D + B \cap C \cap D - A \cap B \cap C \cap D$$

This leads to the following quotient for LCM(a, b, c, d):

$$\frac{abcd\,\mathrm{HCF}(a,b,c)\,\mathrm{HCF}(a,b,d)\,\mathrm{HCF}(a,c,d)\,\mathrm{HCF}(b,c,d)}{\mathrm{HCF}(a,b)\,\mathrm{HCF}(a,c)\,\mathrm{HCF}(a,d)\,\mathrm{HCF}(b,c)\,\mathrm{HCF}(b,d)\,\mathrm{HCF}(c,d)\,\mathrm{HCF}(a,b,c,d)}$$

You cannot get all $2^4 = 16$ possible regions by overlapping four circles, so you need to use ellipses or make drawings such as the one shown in Figure 3.

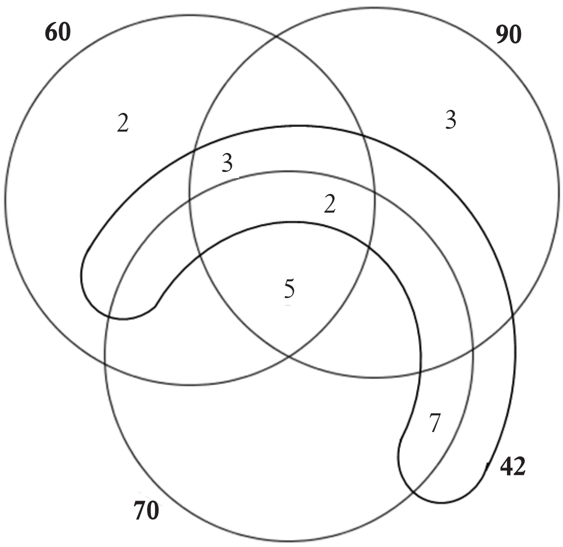


Fig. 3 The numbers 42, 60, 70 and 90

To understand mathematics, learners need to work on tasks which may be considerably different from what Prestage and Perks (2006) call ‘practising the finished product’. A musician does not become an expert at a piece of music by simply sitting down and rehearsing that piece over and over again. They do scales and play other pieces and develop their musicality by listening to other musicians – even vastly different styles of music. Similarly, learners need mathematical tasks that do not simply model an efficient algorithm but which expose the structure of some mathematical area and encourage them to think and make connections between different ideas. Rather than treating HCF and LCM separately, it is much more useful to study them together. By approaching HCF and LCM through puzzles and investigations of this kind, learners can gain practice at performing the procedures while, hopefully, thinking a little more deeply about the concepts.

References

Banwell, C., Saunders, K. and Tahta, D. 1986 *Starting Points*, Tarquin: see ‘Rebounds’, p. 33.

Prestage, S. and Perks, P. 2006 ‘Doing Maths or Practising the Finished Product’. In Hewitt, D. *Proceedings of the British Society for Research into Learning Mathematics,* **26**, 1, February 2006.

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Author Colin Foster, King Henry VIII School, Coventry CV3 6AQ.
e-mail: **c@foster77.co.uk**
website: **www.foster77.co.uk**

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Progress to Advanced Mathematics
Arbelos, UK
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£9 + delivery

Progress to Advanced Mathematics is another from Arbelos along the same lines as *Progress to Higher* except this is written for the transition to the English system. The former book was a success and it would appear this will be too.

This book can be used throughout the fifth year of secondary to consolidate and extend some basic methods. It starts and finishes with chapters on algebra emphasizing the importance of this topic not just in A level or the IB, but in pure mathematics. The chapters begin with straightforward, routine questions and usually end with more challenging questions which are indicated with symbols. Many sections start with the aims of the topic and what you should learn and then ends with miscellaneous questions. There are over 50 exercises with answers at the back.

All the main topics are covered in algebra, geometry and trigonometry and therefore the book provides a full revision/consolidation of the GCSE course (or equivalent). The questions are well set out (not too many on a page) with unambiguous instructions. All diagrams are clear and the grading of questions should give every student confidence with the basics of graph work.

There is also a checklist of learning outcomes which can be downloaded free from the website. This links the book with a sheet that students can tick off learning concepts once they have been mastered. With only 167 pages, it is packed with plenty of questions and being relatively thin and light should have a reasonable life span. At least one class set would enhance any department.

N. G. Macleod

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