Questions Pupils Ask!

Is i irrational?

by **Colin Foster**

I once saw a Key Stage 3 lesson in which pupils were simplifying algebraic expressions, and where each question part (a, b, c, …) involved only the letter of that question, so part b might have been something like $5b - 2b + b - 3b$. When some of the pupils reached part i and were working on a question like 8i + 4i − 9i + 3i, the teacher stopped the class and suggested to the pupils (tongue in cheek, of course) that in that question part they were actually doing something much more advanced. They could congratulate themselves that they were doing complex numbers, since in maths i actually means an imaginary number (Note 1)!

Of course, the teacher was not being totally serious about this, and had chosen to take an opportunity to introduce the pupils to the problem of square-rooting a negative number, saying that we use the letter i as a 'made-up number' for $\sqrt{-1}$. But it got me thinking whether in any sense it could be true to say that they were 'doing complex numbers'? On the face of it, their activity didn't look much different from that of a further mathematics student in their first lesson on complex numbers, simplifying expressions like 3 + 8i − 5 − 2i. Is i + i = 2i only adding imaginary numbers if I think about the fact that i = $\sqrt{-1}$ as I do it? Of course, it is only a convention to use i for $\sqrt{-1}$; any other part of that question could equally well have been thought of as involving complex numbers, as of course nowhere did the book specify that the quantities were all real!

Sometimes we do not always state clearly what we mean. For example, we might say that although we can factorize $x^2 - 1$, as $(x + 1)(x - 1)$, we *can't* factorize $x^2 + 1$, for example. However, just changing the letter name from x to z might suggest otherwise. On seeing $z^2 + 1$, we might think in terms of working over the complex numbers, rather than over the reals, and so say that we *could* factorize it, as $(z + \mathrm{i})(z - \mathrm{i})$ (Note 2). So the answer to "Is $x^2 + 1$ irreducible?" depends on the set of numbers which we consider the coefficients to belong to, and something as slight as a choice of a letter x could push us towards considering only real numbers.

Quite often, statements are not returned to and reformulated as new things are learned. Assumptions build up that may be helpful for a while but become problematic later (Tall, 2013). For example, typically the idea of 'irrational' gets defined long before complex numbers make an appearance. An irrational number is any number which can't be written in the form $\frac{p}{q}$, where p and q are integers. Perhaps we should have said 'any *real* number …'? But then what would that have added for someone who hasn't yet met any number that *isn't* real, or doesn't even know that such a number could exist? It's the sort of thing the teacher might slip in for the sake of their own mathematical conscience, but which is either not noticed by the pupils or leads to questions which are likely to divert significantly from the main point of the lesson. (If you are introducing irrational numbers today, you might not want today also to be the lesson where you have to introduce the idea of an imaginary number.) So one day, years later, when a pupil asks, "Is i irrational?" there may be a bit of work to do in sorting it out.

It may be that this question arises partly because i is the first letter of irrational (as well as imaginary). It might also be provoked by the cartoon (Fig. 1) that occasionally does the rounds among sixth formers, which personifies the numbers i and π, and has i saying to π, "Be rational!" and π saying to i "Get real!" This joke depends on the fact that π is real but not rational, but might be understood as suggesting that i is rational but not real, otherwise i is being hypocritical! So is i rational or irrational?

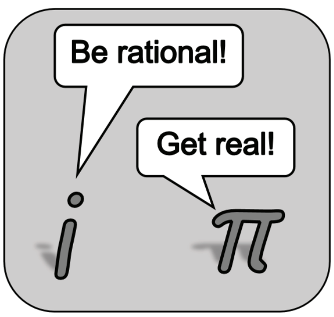

Fig. 1 A number joke

By our definition above, i is not rational, since it cannot be expressed as the ratio of two integers. The only square roots of integers which are rational are the square roots of positive square numbers (and zero). Pupils might think that although $\sqrt{-5}$ is not rational, $\sqrt{-9}$ might possibly be, since 9 is a square number. However, writing $\sqrt{-9} = \frac{p}{q}$, where $q \neq 0$, leads to $p^2 = -9q^2$, which has no solutions for real p and q, except $p = q = 0$, which contradicts the statement that $q \neq 0$. So based on this we cannot call i rational.

So i is not rational — so is it irrational then? Not so fast! 'Irrational' is normally taken to refer to a *real* number which is not rational, so on this basis i is neither rational nor irrational. Just as it doesn't make sense to ask whether $\frac{2}{3}$ is odd or even, because it is neither of the form $2n$ nor of the form $2n + 1$ (where n is an integer), it doesn't make sense to ask whether i is rational or irrational. (Pupils may think of 'odd' as any number that isn't even but not state carefully that 'number' in this context means 'integer'.) Rational and irrational are opposites, so it may seem that a number which isn't one must be the other, and this is true across the reals, just as all integers are either even or odd. However, the symmetry between evens and odds is really quite unlike that between the rationals and irrationals. We can specify what all the rationals are like (fractions with integer numerator and non-zero integer denominator), but the irrationals (of which, in a sense, there are far more) include not just the surds but all kinds of strange things like e and π, and even lots of numbers which we can't express in closed form. The irrationals feel like the contents of a bin into which we put all the badly-behaved numbers – and for this reason it might feel like i really ought to go in that bin!

Another problematic issue that arises with imaginary numbers is to do with ordering. We know that for any two real numbers, a and b, exactly one of the statements

$a = b, a > b, a < b$

must be true (the law of trichotomy). However, we can't do this with complex numbers. They can't be ordered in a useful way. If we take, for instance, 2 + 3i and 3 + 2i, we *can't* say

$$2 + 3i = 3 + 2i$$
$$\text{or } 2 + 3i > 3 + 2i$$
$$\text{or } 2 + 3i < 3 + 2i.$$

None of those three statements is true. If we think of the *modulus* of the two complex numbers, then we *can* say $|2 + 3i| = |3 + 2i|$, since the modulus of a complex number is a real number, and we know that we are OK ordering real numbers.

When at school, I mistakenly thought that this was obvious. The complex plane is 2-dimensional, I thought, so how can you possibly put all those points in order, whereas the real axis is a line, with the points neatly ordered along it. Just thinking of the integers, we have:

$$\ldots -3 < -2 < -1 < 0 < 1 < 2 < 3 \ldots$$

Similarly, the imaginary axis is also a line, so presumably we can write:

$$\ldots -3i < -2i < -i < 0 < i < 2i < 3i \ldots$$

But in general, the non-reals have two parts to them – they are like '2-dimensional numbers', so they can't be put into order along a 1-dimensional line.

However, this is not right at all, as I only realized much later on! The symmetry between the reals and the purely imaginaries is deceptive. Maybe the use of vertical number lines in school (e.g. temperature scales for working with negative numbers) contributes to this easy false idea? But the Argand diagram is not merely a horizontal number line coupled with a vertical number line. The problem is that we cannot say that 3i > 2i. This is quite counterintuitive: surely, whatever i is, 3 lots of it is more than 2 lots of it? But no! Let's suppose that 3i > 2i. If we subtract 2i from both sides, we get i > 0. But that cannot be true. If we multiply both sides of this inequality by i, since we are supposing that i is positive (i > 0), the inequality sign will stay the same way round, giving $i^2 > 0$. But by the definition of i we know that $i^2 = -1$, giving us $-1 > 0$, which is clearly false. This means that we must have been wrong to suppose that i > 0, or that 3i > 2i.

So we might think that, as it came out the wrong way round at the end, with $-1 > 0$ instead of $-1 < 0$, we must just have started the wrong way round at the beginning. We have found that i isn't positive, so it must be negative? Let's try that instead. Starting with i < 0, if we multiply both sides of this inequality by i, then since i is supposed to be *negative* now, this will *reverse* the inequality sign. That means that we get $i^2 > 0$ again, just like before, which we have just shown can't be true!

If i > 0 is false, and i < 0 is also false, perhaps we should try i = 0? This seems unlikely. Starting with i = 0, we can multiply both sides of this equation by i, and we get $i^2 = 0$, which means $-1 = 0$, which is again false, of course. So we must have been wrong to say i = 0.

This means that i is not greater than zero, or less than zero, or equal to zero. We can't use inequality signs between non-real numbers. I think I didn't realize that this applied to the *purely* imaginary numbers until at university I noticed that writing "Let ε > 0" also meant by implication that ε was real. But I should have understood this at school. This raises the question of what the imaginary axis on an Argand diagram is doing? Isn't the imaginary axis an ordering of the purely imaginary numbers? If they really can't be ordered, would it make just as much sense if, instead of numbering the imaginary axis 0, i, 2i, 3i, 4i, etc. (Fig. 2), we numbered it haphazardly, going up

$$-7i, \frac{2}{3}i, 5i, -4976i, \text{etc?}$$

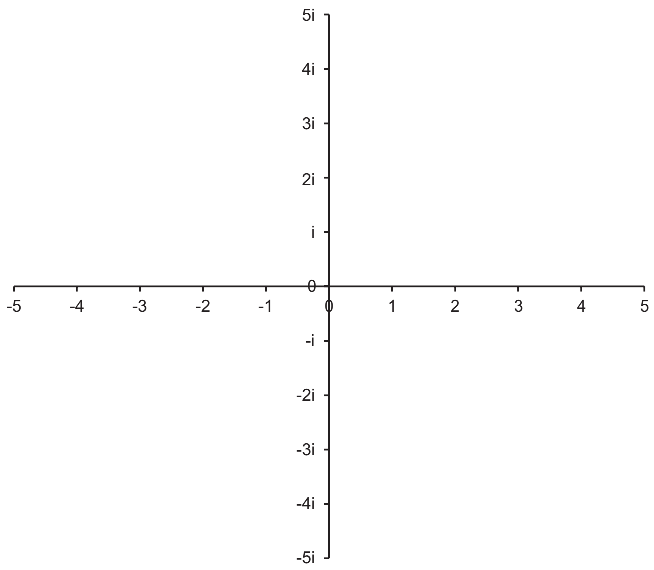

Fig. 2 A common way of labelling the axes in an Argand diagram

Eventually I realized that what we are doing with the imaginary axis on an Argand diagram is plotting the imaginary *parts* of the purely imaginary numbers – and of course those imaginary *parts* are real! So, for a number like 4i, we plot Im(4i) = 4 at (0, 4). So it's actually *real* numbers that we are plotting on the Argand plane, and real numbers can be ordered. The fact that zero lies on the imaginary axis should really have alerted me to this, since 0 is not an imaginary number (even if you write it as 0i)!

Perhaps this means that instead of the labelling as in Figure 2 (which is common in textbooks), we should be a bit more careful and do it as in Figure 3? The complex number $z = x + iy$ is represented by the point (x, y), and of course here both x and y are real. Perhaps this also means that we shouldn't refer to the 'positive imaginary axis' in the Argand diagram? Or, if we label the real axis with an arrow on the right-hand end, to indicate direction, maybe we *shouldn't* do this at the top of the 'imaginary' axis? Although, I suppose, we could say that what we mean by the 'positive imaginary axis' is the part of the imaginary axis relating to imaginary numbers whose *imaginary parts* are positive!

All of this is closely related to the symmetry between the two square roots of –1. We shouldn't call i the 'positive square root' of –1 and –i the 'negative square root' of –1. Some would say that we shouldn't even write $\mathrm{i} = \sqrt{-1}$, because the radical symbol conventionally means 'the positive square root of …'. As we have seen, despite appearances, neither i nor –i is positive or negative, since it makes no sense to say that either of them is greater or less than zero. In fact, neither i nor –i is more fundamental, and we could go through the whole of mathematics replacing all the i's with –i's and everything would work out fine! Note that this is *not* the case with 1 and –1. For example, 1 × 1 = 1 but –1 × –1 ≠ –1. Doing a search-and-replace in all of mathematics and swapping all the 1s for –1s would create a huge mess! Again, I remember thinking at school that the pure imaginaries were just like the reals; that there was a symmetry between the two. I remember thinking that if we had 'discovered imaginary numbers first' we might have called them the reals and children would count their sweets as i, 2i, 3i, … (although no doubt lazily dropping the i's!). I think the apparent two-way symmetry of the Argand diagram contributed to this misconception on my part. But the left–right symmetry of the Argand diagram is quite unlike the top–bottom symmetry, so I really was wrong about quite a lot of things!

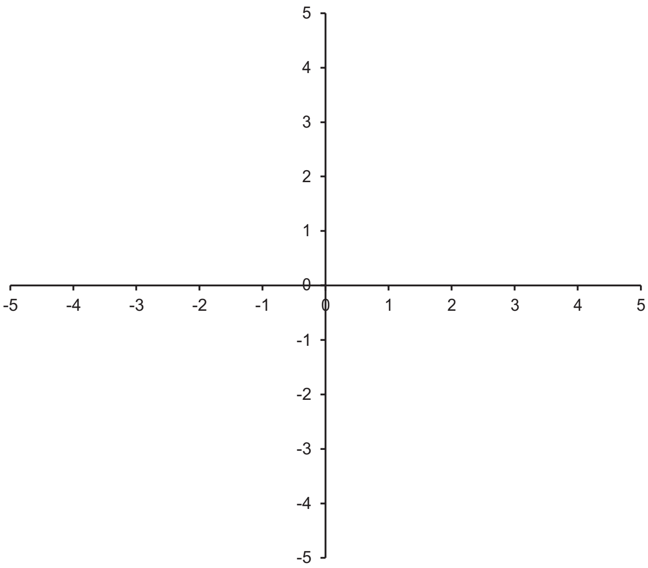

Fig. 3 Labelling the axes of an Argand diagram with the real and imaginary ***parts***

Notes

1. A cynic could argue that this was not 'actually' the case, since the letter *i* was italic, and imaginary i is normally printed non-italic!
2. This is actually a bit subtle, because the issue is not really whether the variable *x* or *z* might be non-real so much as whether the *coefficients* might.

Reference

Tall, D. 2013 *How Humans Learn to Think Mathematically: Exploring the Three Worlds of Mathematics,* Cambridge University Press, Cambridge.

Colin Foster was recently interviewed on the Mr Barton podcast: see **http://www.mrbartonmaths.com/blog/colin-foster-mathematical-etudes-confidence-and-questioning**

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