Questions Pupils Ask!

What is the formula for factorial?

by **Colin Foster**

Sometimes pupils assume that in mathematics there must be a formula for everything (Foster, 2010). When doing investigations, they sometimes ask each other, “Have you found the formula?”, as though there must be one all-embracing formula that will give ‘the answer’. Indeed, this may well be because that is often how they are judged to have been successful. But even though there may be more to working on a task than finding ‘the formula’, it is nonetheless true that often in mathematics there are beautifully neat generalizations that capture something seemingly complex in a short, simple form.

The summation of the first n positive integers, usually attributed to Gauss, is a good example. My sixth form students had been working on sequences and series, and had enjoyed Gauss’s clever trick of summing the integers from 1 to 100 by writing them out twice

$$\begin{array}{ccccccccccccc} 1 & + & 2 & + & 3 & + & ... & + & 98 & + & 99 & + & 100 \\ 100 & + & 99 & + & 98 & + & ... & + & 3 & + & 2 & + & 1 \end{array}$$

and noticing that each of the 100 vertical pairs adds up to 101, meaning that the total must be $\frac{1}{2} \times 100 \times 101 = 5050$. This had led to us generalizing

$$\sum_{k=1}^{n} k = 1+2+3+\cdots+n = \frac{1}{2}n(n+1),$$

which seemed very neat.

This playing around with consecutive integers reminded one of the students of factorials, and he asked about *products* of integers, which I said could be expressed similarly as

$$\prod_{k=1}^{n} k = 1.2.3\cdots n.$$

But what does *this* equal?

Well, of course we can say that it is equal to n factorial and write

$$\prod_{k=1}^{n} k = n!$$

and then use our calculators, but inventing a new symbol like ‘factorial’ feels like an admission of defeat! It would be like writing

$$\sum_{k=1}^{n} k = 1+2+3+\cdots+n = \Delta_n,$$

and just saying that Δ_n means the nth triangle number, without finding a formula for it. Can’t we come up with a simple formula for factorial like we did for the *sum* of the integers?

By analogy with the sum formula, the product formula should surely be something like $\sqrt{n^{n+1}}$? We quickly disproved the validity of guesses such as this by establishing that most of them rarely even give integer answers, let alone the correct value of n! However, students felt sure that a similar trick to Gauss’s should work – whereas I had that feeling that ‘if this were possible I would already know about it’! They wrote:

$$\begin{array}{ccccccccccccc} 1 & \times & 2 & \times & 3 & \times & ... & \times & 98 & \times & 99 & \times & 100 \\ 100 & \times & 99 & \times & 98 & \times & ... & \times & 3 & \times & 2 & \times & 1 \end{array}$$

and were disappointed that this didn’t seem to help, as the products of the vertical pairs were not the same.

Perhaps a 100 × 100 Latin square would be more useful?

$$\begin{array}{ccccccccccccc} 1 & \times & 2 & \times & 3 & \times & ... & \times & 98 & \times & 99 & \times & 100 \\ 2 & \times & 3 & \times & 4 & \times & ... & \times & 99 & \times & 100 & \times & 1 \\ 3 & \times & 4 & \times & 5 & \times & ... & \times & 100 & \times & 1 & \times & 2 \\ ... & \times & ... & \times & ... & \times & ... & \times & ... & \times & ... & \times & ... \\ 100 & \times & 1 & \times & 2 & \times & ... & \times & 97 & \times & 98 & \times & 99 \end{array}$$

Now each row and each column has a product of 100! If we add up the columns we get 100(100!), and this must be equal to the rows, which also add up to 100(100!). True, but not very helpful!

We know that a variation on Gauss's sort of approach works for summing geometric sequences too, where we write:

$$
\begin{array}{lccccccccccccc}
S = & a & + & ar & + & ar^2 & + & \dots & + & ar^{n-2} & + & ar^{n-1} & & \\
rS = & & & ar & + & ar^2 & + & ar^3 & + & \dots & + & ar^{n-1} & + & ar^n.
\end{array}
$$

So $\quad rS - S = ar^n - a,$

giving $\quad S = \dfrac{a(r^n - 1)}{r - 1}, \quad r \neq 1.$

It really feels as though we ought to be able to do some sort of trick like this with factorials. They are constructed in such a regular way – multiplying consecutive integers – that the students felt that there really ought to be a simple formula for them. Soon I was being asked, "So is it not possible then?" and "Why not?" and "Do you actually *know* it's not possible or is it just that no one's come up with a formula for it yet?"

I offered the recursive formula $n! = n(n - 1)!$, for integer $n \geq 1$, but, unsurprisingly, they were not impressed with that. Inductive definitions rarely feel like 'proper' formulae to students – "That's basically just telling you to work it out the long way!" – like saying $\Delta_n = \Delta_{n-1} + n$. Naturally, they wanted an *explicit* formula, where you put in n and it gives you $n!$, without any small-print about having to work out lots of other factorials beforehand!

Well, $n!$ can be expressed as $n! = \int_0^\infty t^n e^{-t} \mathrm{d}t$ for natural n, but why are we getting into such advanced mathematics just to work out the product of consecutive whole numbers? There is Stirling's approximation, $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$ for large n, and we just want an exact version of that kind of thing.

This would appear to be something to add to my list of 'things that seem to be much harder than they ought to be'. We can come up with simple formulae for the sums of sequences created by adding a constant difference (arithmetic sequences) or multiplying by a constant factor (geometric sequences), but we don't seem to be able to find a simple formula for the *product* of even the *simplest* (add 1) finite sequence. Why should products be so much harder than sums? My response to the students that it all depends on what you count as 'a formula' fell a little flat, as we all basically knew that we wanted something nice and neat involving n. Can we show that this is impossible, or suggest why that might be? I would be very interested if any reader can give a better response than I was able to.

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References

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