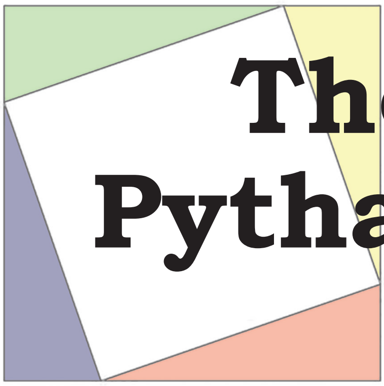
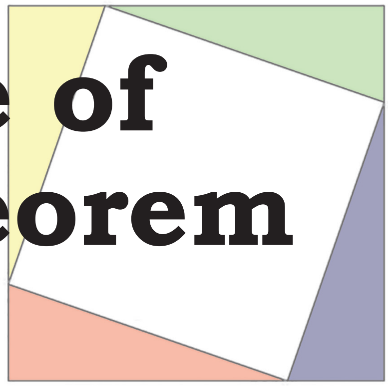


The Converse of Pythagoras' Theorem



by **Colin Foster**

People often muddle up the words *inverse* and *converse*, but they are not interchangeable. For example:

Statement A: 'Any multiple of 4 is even' is true.

The *converse* exchanges the premise and the conclusion:

Converse of A: 'Any even number is a multiple of 4', which, in this case, is false – for example, 6 is even but not a multiple of 4.

The *inverse* of the original statement takes the negation of the premise and the negation of the conclusion, but keeps them in the same order:

Inverse of A: 'Any number that is not a multiple of 4 is odd'.

In this case, this is also false – for example, 6 is not a multiple of 4, but it is not odd.

There is also the *contrapositive* of the original statement, which negates both the premise and the conclusion, and also swaps the order:

Contrapositive of A: 'Any number that is odd is not a multiple of 4'.

This is true. (A statement and its contrapositive are always either both true or both false.) Note that the inverse is the contrapositive of the converse (Note 1). So, starting with A ⇒ B, its *converse* is B ⇒ A, its *inverse* is A ⇒ B' and its *contrapositive* is B' ⇒ A'.

Why does this matter? Well, the converse of a true theorem is sometimes true and sometimes false. I was thinking about this recently while watching a lesson on Pythagoras' Theorem. The converse of Pythagoras' Theorem says that, if the square of two sides of a triangle sum to the square of the third side, then the triangle is right-angled (and the right angle is in between those first two sides). I wonder how often this gets proved in lessons at school, or even how often the *need* for its proof is realized by pupils?

In the lesson I was watching, the teacher had introduced Pythagoras' Theorem as a fact (without any proof). It was offered as a 'rule', maybe like a law in science, with no discussion of whether it was exact or an approximation. I began to wonder why this happens – maybe the teacher was intending to work on a proof later on? The teacher told me afterwards that she felt that a proof would have been too difficult for the class, because they were not confident with expanding brackets. The proof that the teacher had in mind was the one shown in Figure 1.

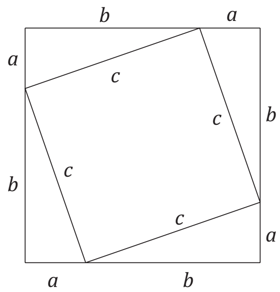


Fig. 1

By equating the area of the large square, calculated in two different ways,

$$(a+b)^2 = 4\left(\frac{1}{2}ab\right)+c^2$$
$$a^2+2ab+b^2 = 2ab+c^2$$
$$a^2+b^2 = c^2.$$

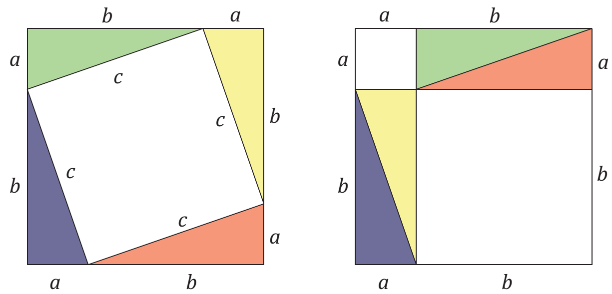


Fig. 2

But it is possible to do essentially the same proof without any algebra at all, provided we are happy that translations preserve area. In Figure 2, we simply translate the right-angled triangles. We can see at a glance that the white area that is c^2 in the left-hand drawing is now rearranged into a^2 (the top left square) and b^2 (the bottom right square) in the right-hand drawing. To make this rigorous, we would have to be sure that the white shape in the left-hand diagram is really a square, but really we should check things like that in the Figure 1 proof too.

This got me thinking about whether a simpler proof is necessarily a better proof. Should we always try to prove things in the way that we think students will find easiest (Foster, 2008, 2016)? Perhaps, if we simply want to get past the proof, and onto 'using' the idea, then the most streamlined proof we can find may be best. And, certainly, a proof doesn't have to have algebra in it for it to be a 'proper' proof. But, on the other hand, you might want to take the opportunity here to practise expanding brackets, and to see a use for that technique. And it only entails adding the lines shown in Figure 3 to reveal the structure of $(a + b)^2 = a^2 + ab + ab + b^2$ within the diagram, so perhaps it is a missed opportunity to deliberately avoid this?

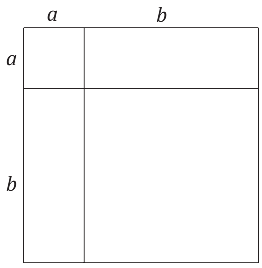


Fig. 3

There is also a case that some proofs are 'canonical', perhaps including Euclid's more difficult proof of Pythagoras' Theorem, and we may argue that we want students to experience them, even if they may be harder to grasp than some other proof. And, of course, sometimes the whole point is the particular method of proof, rather than the result, so it all depends on the teacher's purposes.

Back in the lesson I was watching, pupils were working on exercises from a sheet of paper, each one involving finding the missing side of right-angled triangles where two sides were given. The final question on the sheet was causing a bit of discussion:

A triangle has sides of length 6 cm, 8 cm and 10 cm. What can you say about the triangle?

Pupils were stumped by this different style of question. It was the only question without a little sketch drawing of the triangle. All three lengths were given, so what could possibly be left for them to work out? The fact that all the triangles in today's lesson were right-angled had not been highlighted particularly.

Eventually, a pupil made the connection and said that the triangle must be right-angled, because "Pythagoras' Theorem is true", by which he meant that $6^2 + 8^2 = 10^2$ (Note 2). The teacher was very pleased about this, and said to me afterwards that she was delighted that the pupils had "made the connection for themselves" and had "worked backwards" in this way. She saw this question as a kind of "inverse problem", because the students had "the answer" and were working back to "the question". She saw this as a nice twist – a bit like ending a sheet of 'Calculate the area and perimeter' questions with "A rectangle has a perimeter of 24 cm; what could its area be?"

But this is problematic, because the question could only be answered by using the *converse* of Pythagoras' Theorem, and the students had never been even told that the converse of Pythagoras' Theorem was true, let alone proved it. Perhaps we should have been more pleased if the pupils had answered the question, "What can you say about the triangle?" by answering "Absolutely nothing"! Perhaps I am being too awkward, but it seems to me that it is a problem if pupils are encouraged to assume that if $A \Rightarrow B$ it must follow "logically" that $B \Rightarrow A$. That is not the case. Often, the converse of a theorem *is* also true, but not always, and certainly not necessarily. The converse of Pythagoras' Theorem *is* true, but that has to be thought about and proved, not just assumed.

What we have to do to prove the converse of Pythagoras' Theorem is to take a triangle with sides a, b and c, such that $a^2 + b^2 = c^2$. Then we construct *another* triangle with sides of length a and b, *containing a right angle*. By Pythagoras' Theorem, we know that the hypotenuse of this *second* triangle must have length $\sqrt{a^2+b^2}$, which (because $a^2 + b^2 = c^2$) is the same as the length of side c of the first triangle. Since both triangles therefore have the same three side lengths, a, b and c, the triangles are congruent (SSS), and so have the same angles, which means that the angle between sides a and b in the first triangle must also be 90°. It doesn't involve any hard algebra, but to understand what is going on, and what it really shows, requires a bit of thought. Is this worth spending time on? Or is it a lot of fuss about nothing?

If we are pleased when pupils assume that the converse of a theorem must be true, we will be sorry later on when we have to convince them that this is not always the case. It is not 'future-proofing' our teaching. When I mentioned this to the teacher, her response was to point out that we know from Pythagoras' Theorem that a 3-4-5 triangle is right-angled, so we also know that a 6-8-10 triangle must be right-angled too, as it's just a scaled-up version. This is a bit subtle. Pythagoras' Theorem tells us that *if we know that a triangle is right-angled*, and its legs are 3 and 4, then we know that its hypotenuse is 5. So, there *is* a 3-4-5 triangle that is right-angled. But Pythagoras' Theorem by itself doesn't tell us whether there might be *other* 3-4-5 triangles that are not right-angled (Note 3). We have to combine Pythagoras' Theorem with the (obvious? intuitive?) knowledge that a triangle with fixed side lengths is *rigid* (i.e. congruence by side-side-side), and can't be wobbled around (unlike a *quadrilateral* with fixed side lengths, which can), in order to say that *all* 3-4-5 triangles are similar, and therefore right-angled. (Or, disregarding units, we would say that there is just one 3-4-5 triangle.)

Maybe this feels so obvious that it doesn't need to be said? Perhaps, when the converse follows so intuitively, we should not make a big fuss about it – and perhaps we should then include the converse within our statement of 'Pythagoras' Theorem', as some do (e.g. Alcock, 2017). But a converse of a theorem may seem obvious until you remember that very often the converse of a theorem is actually false.

Notes

1. For all these examples, we assume that we are dealing with integers, otherwise a number could be neither odd nor even.
2. Of course, Pythagoras' Theorem is *always* true, otherwise it wouldn't be a theorem. It's not that the theorem "goes wrong" for non-right-angled triangles; the Theorem makes a statement that is conditional on the triangle being right-angled!
3. Unless we include the converse of Pythagoras' Theorem in our statement of "Pythagoras' Theorem", but it wasn't presented to the pupils in that way at the start of the lesson I watched.

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