THIRTY FACTORS

30 **30** 30

by **Colin Foster**

Can you write down a number that has exactly 30 factors? Is that an easy or a hard thing to do? What kinds of numbers have exactly 30 factors? What is the *smallest* number with exactly 30 factors? If you can't immediately see how to tackle this last question, you might like to estimate roughly how big you expect the answer to be.

Exploring how many factors different numbers have is one of my favourite investigations. I have used it in some form with all ages from Year 7 to sixth form and with PGCE students and experienced teachers. I was recently visiting schools in Uganda to observe mathematics lessons, and arrived at one school ready with my empty notebook only to find myself directed to the school hall where about 300 Senior 5 and 6 students (A level) were waiting for me. I had 90 minutes and complete freedom, so I quickly had to think of some mathematics to do with them! So I began with this task. There are different ways to introduce it: asking for the smallest number with exactly 30 factors is challenging, but then you can encourage students to tackle the problem by posing their own simpler questions to begin with, such as reducing 30 to something like 4 to start with, and perhaps not worrying about finding the *smallest* number initially.

A gentler way to begin is:

How many factors does 10 have?

Although this is still a closed question, it requires slightly more work than "What are the factors of 10?" and focuses attention on the *number* of factors rather than on what they are. Students might need to agree that 1 and 10 should be included. Naturally once we have the answer that is only the beginning – ultimately we want to ask "*Why* should 10 have four factors?" – so you can follow this up with:

Find some more numbers with exactly 4 factors.

Subsequently we will want to categorize numbers with *any* number of factors. But starting with 4 can be helpful because it is the first slightly more complicated case, since there are *two kinds* of number that have four factors. This emerges if you make a list as follows:

Number of factors	Numbers
1	
2	
3	
4	
...	

The 'numbers' column can contain examples or (perhaps later on) generalizations. The only number with exactly 1 factor is 1, so the first line is easy. Students will realize that numbers with two factors are prime, but will probably not know anything in particular for later lines, so these can be left blank for now. Finding numbers with exactly 3 factors usually leads to a conjecture (i.e. somebody shouts it out!) that they are square numbers, and this is a nice opportunity to consider necessary and sufficient statements:

n has 3 factors $\Rightarrow n$ is square but n is square $\not\Rightarrow n$ has 3 factors.

In general, square numbers have an *odd* number of factors, but not necessarily 3. Someone will find a counterexample such as 16 or 36 to the conjecture that all square numbers have 3 factors. So why do square numbers have to have an odd number of factors? Students usually work out why if they try to list the factors systematically:

e.g. 36: 1, 2, 3, 4, **6**, 9, 12, 18, 36

Most of the factors come in pairs, but then 6 pairs with itself ($6 \times 6 = 36$), but it's the same number, so we count it only once. So it is fairly convincing from this that square numbers will always have an odd number of factors, and that the only numbers with an odd number of factors will be the squares, because otherwise the factors will come in pairs, and there will be an even number of them.

Eventually somebody will notice that the numbers with exactly 3 factors are the *squares of primes*, because if p is a prime then the factors of p^2 must be 1, p and p^2. There can't be any more factors, because the only factors of p are 1 and p, so the only factors of p^2 will be 1, p and p^2.

So what about numbers with exactly 4 factors? The first few such numbers are: 6, 8, 10, 14, 15, 21, 22, 26, 27, 33, 34, 35, 38, 39, … and it can be hard to see what is going on because we have two different kinds of number here. Numbers with exactly 4 factors are always either of the form p^3 or of the form pq, where p and q are (distinct) primes, so they are either the cubes of a prime number or are *semiprime* (i.e. the product of two distinct primes). We can justify this by writing out the factors of each in order:

p^3: $1, p, p^2, p^3$

pq: $1, p, q, pq$

(assuming, without loss of generality, that $q > p$).

You can see here what would happen with pq if p and q were equal: we would end up with $1, p, p, p^2$, which is just *three* factors: 1, p and p^2. So we would be back to the 3-factor case for the square of a prime.

It may now be clear that p^n is always going to have $n + 1$ factors. We can think of 1 as being p^0, so there are $n + 1$ possible powers of p, including the zeroth power. So going back to our original problem we can immediately say that 2^{29} would be a number with 30 factors. This is very powerful, because we might have no idea of the numerical value of 2^{29}, and yet we can say with certainty that it has 30 factors. In fact if we were given the numerical value of 2^{29}, which is 536 870 912, it would be much harder to realize that it had 30 factors than if we saw it written as 2^{29}. Students often want to 'work out' numbers written as powers, but index form can be much more revealing, as we can see their structure.

So we know that 2^{29} will certainly have 30 factors, but is it the *smallest* such number? Students often think so, because they see the only alternatives as 3^{29}, 5^{29}, 7^{29}, and so on, and since 2 is the smallest prime then 2^{29} should be the smallest number with 30 factors. But we saw that there was another way besides p^3 of making numbers with 4 factors, so should there not also be other ways of making numbers with 30 factors? We wrote out the four factors of pq to see why there were four factors, but we can examine this more carefully. The two distinct prime numbers p and q can each appear as either of two possible powers (p^0 or p^1; q^0 or q^1), and that explains why we have four factors for numbers of the form pq: we have 2×2 factors altogether. So we can see that there are two ways of making numbers that have exactly 4 factors because 4 itself has two factor pairs:

$4 = 1 \times 4$ corresponds to p^3

$4 = 2 \times 2$ corresponds to pq.

(That is why 4 is the first number for which this happens.) So if the question had asked for the smallest number with *31* factors (31 being prime), then the answer *would* have been 2^{30}. Although 31 is larger than 30, answering the question for 31 is much easier than answering the question for 30, because 30 has more factors than 31.

So, in general, a number which has a prime factorization $p^a q^b r^c s^d$..., where p, q, r, ... are distinct primes and a, b, c, ... are non-negative integers, is going to have $(a + 1)(b + 1)(c + 1)(d + 1)$... factors, because the prime p can appear as any of its $(a + 1)$ different powers, and so on with each prime.

This is a lovely result, which allows you to easily generate numbers with any desired number of factors. It also allows you to explain interesting patterns. For example,

How many factors does 10 have?

How many factors does 100 have?

How many factors does 1 000 have?

How many factors does 10 000 have?

(A series of closed questions can be much more interesting than any single closed question by itself!)

Did the pattern surprise you? Can you see why it must happen? Students can work out how many factors the numbers in these sequences have and explain the patterns that they find:

2, 4, 8, 16, 32, ...

5, 50, 500, 5000, 50 000, ...

12, 120, 1 200, 12 000, 120 000, ...

1, 4, 9, 16, 25, ...

So to return to our original question, we actually need to think about the number of factors that 30 itself has. This is perhaps quite surprising, since 30 was never the number that we were supposed to be factorizing! We can prime factorize $30 = 2 \times 3 \times 5$, and see that any number with 30 factors must take one of these five forms:

Number of factors	General form	Smallest example
30	a^{29}	2^{29}
15×2	$b^{14} \times a$	$2^{14} \times 3$
10×3	$b^9 \times a^2$	$2^9 \times 3^2$
6×5	$b^5 \times a^4$	$2^5 \times 3^4$
$5 \times 3 \times 2$	$c^4 \times b^2 \times a$	$2^4 \times 3^2 \times 5$

In the third column, for each row we order our primes from smallest to largest as the indices go from largest to smallest, in order to minimize the products. It is fairly easy to justify without working them out that the values get smaller as you go down the table if you think about what is changing from each row to the one below. For example, as $2^{15} > 3$, we know that $2^{29} > 2^{14} \times 3$, since moving from the first row to the second row entails multiplying by 3 and dividing by 2^{15}. Similarly, on going from the third row to the fourth row, since $2^4 > 3^2$, we can see that $2^9 \times 3^2 > 2^5 \times 3^4$, and so on. This means that the smallest number with exactly 30 factors must come from the bottom row, and is $2^4 \times 3^2 \times 5 = 720$. Was that about the size of number you were expecting?

The smallest number with n factors for different values of n makes an interesting object of study. If we list the smallest numbers with 1, 2, 3, ... factors, we get the sequence 1, 2, 4, 6, 16, 12, 64, 24, 36, 48, 1024, 60, … [1] and we can notice various interesting things. For example, all the numbers after the first are even, the numbers are alternately square and non-square, and we get powers of 2 at every prime position – and all of this can be explained by things we have discovered along the way.

Note

[1] This is sequence A005179 at **https://oeis.org/A005179**.

Keywords: Conjectures; Factors; Indices; Powers; Primes; Proof.

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