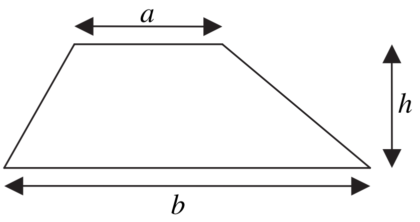
Trapezium Artist: Some thoughts on the formula for the area of a trapezium

by Colin Foster

$$\text{Area} = \tfrac{1}{2}(a+b)h$$



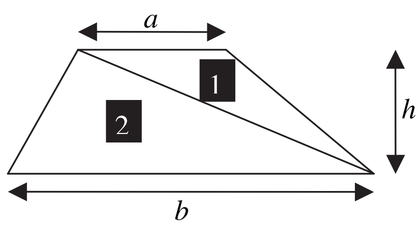
What I thought would be a routine lesson on area with my year 8 class developed in an interesting way. I had given the class a sheet of reasonably random triangles and quadrilaterals and asked them to find the areas by any means they liked. We then discussed their different methods, summarized possible formulas and thought about how to prove them.

The formula for the trapezium stood out as being the only one that wasn't immediately 'see-able'. With thought, those for the triangle, rectangle, parallelogram and kite could all be seen to be correct at a glance. That got us thinking about different ways of proving the formula. We were seeking something not only believable but striking enough as an image to stick in our minds.

All our methods involved converting to simpler shapes.

1. A diagonal line splits the trapezium into two triangles with the same height but different bases.

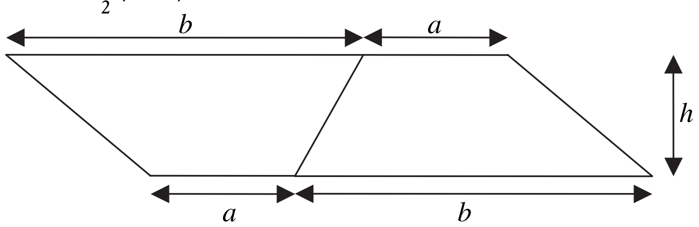
$$\text{Area } 1 = \tfrac{1}{2}ah$$
$$\text{Area } 2 = \tfrac{1}{2}bh$$
$$\text{Area} = \tfrac{1}{2}ah + \tfrac{1}{2}bh = \tfrac{1}{2}(a+b)h$$



This was my preferred proof before the lesson, but the $(a + b)$ element is not visually obvious.

2. Two copies of the trapezium (one upside down) make a parallelogram.

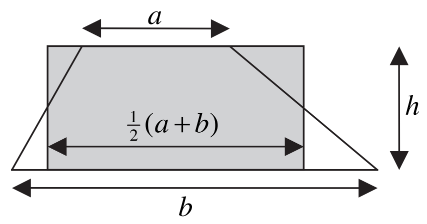
$$\text{Area of parallelogram} = (a+b)h$$
$$\text{Area} = \tfrac{1}{2}(a+b)h$$



Here the $(a + b)$ is clear, but some pupils weren't happy in their minds that the a-side and the b-side would join to make a straight line.

3. A rectangle with the same area and the same height will have a base of $\frac{1}{2}(a + b)$ (the mean of the two horizontal sides).

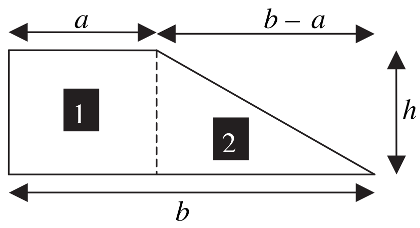
$$\text{Area} = \tfrac{1}{2}(a+b)h$$



This image seemed to fit the formula best, but we felt vague about the locations of the vertices of the rectangle.

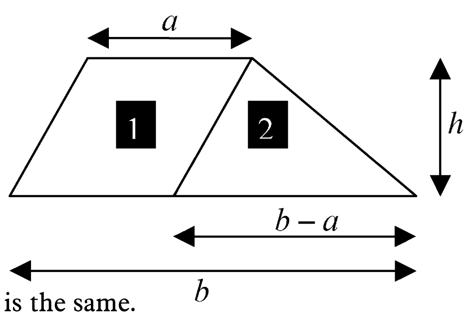
4. We were ready to accept that a horizontal shear would not affect the area. (We had pushed over a stack of file paper and looked at the end when thinking about why all parallelograms with the same base and the same height have the same area.) So any trapezium can be transformed into one with two right angles and the same values of a, b, h and Area.

$$\text{Area } 1 = ah$$
$$\text{Area } 2 = \tfrac{1}{2}(b-a)h$$
$$\text{Area} = ah + \tfrac{1}{2}(b-a)h = ah + \tfrac{1}{2}bh - \tfrac{1}{2}ah = \tfrac{1}{2}ah + \tfrac{1}{2}bh = \tfrac{1}{2}(a+b)h$$



This was a new one for me, and rather neat, but hopeless for visualizing the formula.

It is actually possible to do this without the shear by slicing off a parallelogram from one end of the trapezium (see below).



The algebra is the same.

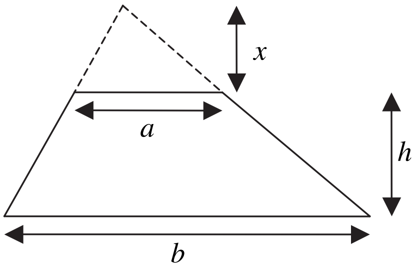
5. A trapezium can be thought of as a triangle with one of its vertices sliced off by a line parallel to the opposite side.

The first step is to find the height x of the small triangle. Since the small triangle is similar to the large one,

$$\frac{x}{a} = \frac{x+h}{b}$$

$$bx = ax + ah$$

$$x = \frac{ah}{b-a}$$



Then,

$$\text{Area} = \tfrac{1}{2}b(x+h) - \tfrac{1}{2}ax$$

$$= \tfrac{1}{2}b\left(\frac{ah}{b-a} + h\right) - \tfrac{1}{2}a\left(\frac{ah}{b-a}\right)$$

$$= \frac{abh + bh(b-a) - a^2h}{2(b-a)}$$

$$= \frac{(b^2 - a^2)h}{2(b-a)} = \frac{(b-a)(b+a)h}{2(b-a)}$$

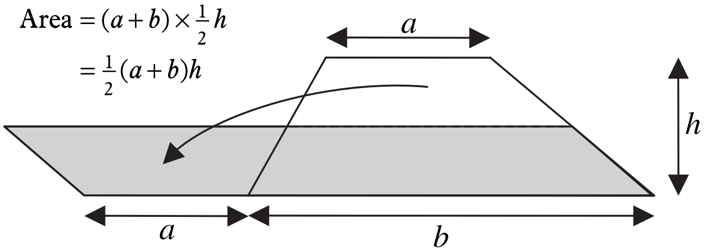
$$= \tfrac{1}{2}(a+b)h$$

I worked on this after the lesson. The algebra was beyond the class. It didn't give me any insight.

6. A final possibility is to slice the trapezium half way up parallel to the base. Rotating the top piece half a turn, it makes a parallelogram with the bottom piece.

$$\text{Area} = (a+b) \times \tfrac{1}{2}h$$

$$= \tfrac{1}{2}(a+b)h$$



This has the advantage over Method 2 that only one copy of the shape is needed.

No doubt one could continue finding other ways. I have ended up quite keen on Method 6.

A Footnote

Later in the lesson, one pupil was staring into space. I asked him what he was doing. He said, "A parallelogram is just a special sort of trapezium, isn't it?" I agreed. "Then the formula for the trapezium ought to work for a parallelogram too." He had tried putting $a = b$ into the formula for a trapezium.

$$\text{Area} = \tfrac{1}{2}(a+b)h$$

$$= \tfrac{1}{2}(b+b)h$$

$$= \frac{2bh}{2}$$

$$= bh$$

(He had made a mistake with his algebra and thought it didn't work – and I was pleased that he was disturbed about that.) We imagined stretching the a-side until it was as long as b, and we could see how the two triangles in Method 1 (our preferred proof) became congruent. We also saw together that putting $a = 0$ gave the formula $\frac{1}{2}bh$ for a triangle and imagined how that would alter our proof (triangle 1 disappearing).

I wish I had thought of all that!

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