Trusting in patterns

By **Colin Foster**

Human beings are natural "pattern sniffers" (Cuoco, Goldenberg, & Mark, 1996), and a sensitivity to patterns is nowhere more critical than in the study of mathematics. Indeed, mathematics is sometimes *defined* as the study of patterns (e.g., Devlin, 1994). But, this attention to patterns has to be more than mindless 'pattern spotting' that fails to engage with the deep structure of the mathematics (Hewitt, 1994). Some tempting mathematical patterns don't continue in obvious ways, such as the sequence

1, 2, 4, 8, 16, *31*, …

that results from counting the number of regions created when marking n points evenly around the circumference of a circle and joining every point to every other point (see Foster, 2007, 2019). (For another nice example of this kind of thing, consider the number of factors of n!) The recent rise in interest in variation theory has encouraged mathematics teachers to make productive use of patterns (e.g., see Watson, Wake & Rycroft-Smith, 2019), but I think there are some dangers with relying on mathematical patterns.

Seductive patterns can lead us astray. Consider these 'trick questions' that children often enjoy:

1. If a white house is made of white bricks, and a red house is made of red bricks, what is a greenhouse made of?
2. If you put tea in a teapot, and coffee in a coffeepot, what do you put in a toaster?
3. If sitting in the bath is a 'soak', and a funny story is a 'joke', what is the white part of an egg called?
4. Say 'silk' five times. Now, what does a cow drink?

Can this sort of thing happen in mathematics? I recently saw a lesson where a pupil had written $3 \times (-6) = 18$. The teacher responded by saying, "Three times *positive* six is *positive* 18, so what must three times *negative* six be?" The pupil, perhaps hearing the emphasised word 'negative', answered, 'Negative 18?' The teacher was pleased – he had avoided telling the pupil the correct answer, or giving her an arbitrary rule; instead, he had encouraged her to rely on a mathematical pattern. This must be good, surely? The pupil has been given not just the right answer to one particular question but a strategy that she can apply across mathematics whenever she is unsure.

But, imagine instead this scenario. No teacher would do this, but imagine if a teacher had said: "If *positive* 3 times *positive* 6 equals *positive* 18, what must *negative* 3 times *negative* 6 equal?" I haven't tried this myself, for obvious reasons, but I think you could get lots of pupils to answer "Negative 18" to this. The language is just as seductive as in those riddles, and you are led to an 'obvious' answer, but this happens to be a pattern that is at odds with the mathematical structure – and yet it sounds just as good. This makes me question how useful these sorts of 'patterns' are, if some patterns are to be trusted and others not. How is the pupil to know, unless they are being led by the nose by the teacher, which patterns you can trust and which ones you can't? I'm also not really sure that this sort of response makes much sense as an argument: $3 \times (-6)$ can't be 18, because some other product, 3×6, is 18. Couldn't you similarly (but wrongly) say: $3 \times (-6)$ can't be -18, because some other product, $(-3) \times 6$ is -18? Why is that not equally valid? Do we have to suppose that the pupil is on board with commutativity of multiplication, but not with 'sign rules'? In the same sort of way, couldn't we conclude that $(-3)^2$ can't be 9, because $(+3)^2 = 9$? Or that 3×0 can't equal zero because 2×0 equals 0? I am not sure where these comparisons end.

I have seen similar teacher interventions in cases where a pupil has written something like $16 \div ½ = 8$. The teacher says: "Sixteen divided by *two* is eight, so sixteen divided by a half can't be 8!" The teacher is attempting to create some cognitive conflict by suggesting that the pupil has what Mason (2017) calls "a correct answer to a different question" (p. 6). This feels like a positive strategy, because, in a sense, you're validating what the pupil is giving you, rather than rejecting it. I particularly like Mason's 'Reversal' strategy for these sorts of situations, which involves asking the question that they have just answered [1]. For example, if the pupil says $2^3 = 6$, then the teacher asks, "What is 2 multiplied by 3?", and the pupil self-corrects, often saying something like, "Oh, that's 6, so 2 cubed must be 8". This seems very helpful if the pupil in some sense 'really' knows the right answer, and has merely made a slip. It's a nudge towards self-correction. On the other hand, if the pupil genuinely believes that $16 \div ½ = 8$, then I am less convinced that an analogy with $16 \div 2$ is necessarily helpful. It seems to rely on an assumption that $a \div b$ can never be equal to $a \div \frac{1}{b}$, which, of course, is false. In fact, these are equal if $b = \pm 1$. Perhaps the hoped-for assumption is that $a \div b$ can never be equal to $a \div c$, if b and c are not equal? But then that isn't true either, if $a = 0$. So, we seem to be relying on pupils knowing that $a \div b$ is never equal to $a \div c$ unless either $a = 0$ or $b = c$. Do pupils really know this? It seems quite complicated!

One of the most popular approaches to the endlessly-discussed example of 'two negatives make a positive' (see Foster, 2015) seems to be to fall back on patterns. If we want to justify the answer to the question "What is 3 – (–1) ?", then one way is to set this up with a carefully-constructed pattern:

$$\begin{aligned} 3 - (+3) &= 0 \\ 3 - (+2) &= 1 \\ 3 - (+1) &= 2 \\ 3 - 0 &= 3 \\ 3 - (-1) &= ? \end{aligned}$$

Following this sequence, it is quite likely that the pupil will agree with us that 4 is 'obviously' the answer, but we have led them into this with a strong steer. However, consider a different question: What is $(-1)^2$? Here, the pupil might look at this pattern:

$3^2 = 9,\ 2^2 = 4,\ 1^2 = 1,\ 0^2 = 0,\ (-1)^2 = ?$

It's much less obvious here what the right answer is, and it might be quite natural to continue:

$(-1)^2 = -1,\ (-2)^2 = -4.$

Isn't continuing downwards just as plausible – or more plausible – than going back up again? Expecting a particular answer to this pattern feels a bit like saying that given the quadratic curve shown in Fig. 1, one of the functions shown in Fig. 2 is 'correct' and the other is 'incorrect'. Isn't this a misconception about what a function must be like? Who is to say that the graph of $y = x^2$ is symmetrical in the y-axis (i.e., is even, as in Fig. 2a) rather than has rotational symmetry of order 2 about the origin (i.e., is odd, as in Fig. 2b)? Who is to say that the 'right' way to continue Fig. 3 is as Fig. 4a rather than Fig. 4b? All of these graphs are perfectly reasonable functions representing perfectly reasonable 'patterns'.

I think a subversive mathematics teacher could easily convince many pupils of wrong answers by strategic (mis) use of patterns. Of course, no teacher would do this – it wouldn't be 'helpful' – but if patterns are only helpful if you know in advance that they are helpful, then that isn't very helpful! The teacher may be able to choose the 'right' patterns, but then this does not seem to be a powerful tactic for pupils who aren't already sure of the answers.

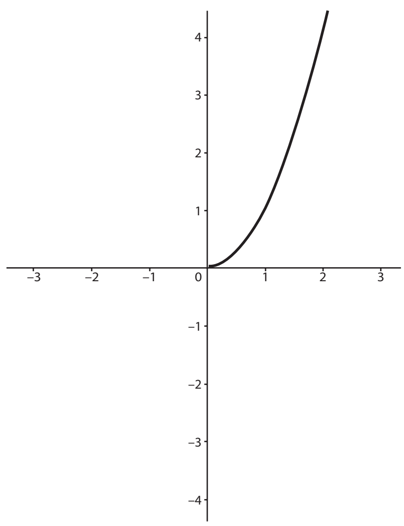

Fig. 1. $y = x^2,\ x > 0$

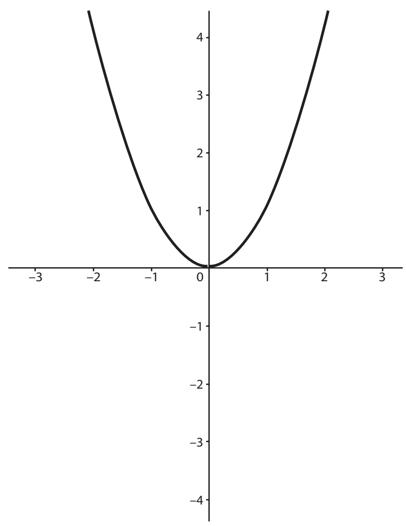

Fig. 2(a) $y = x^2$

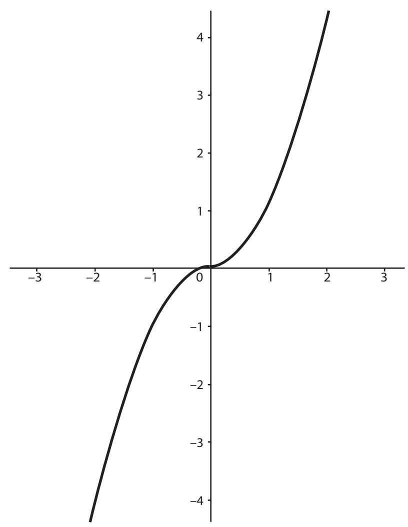

Fig. 2(b) $y = x|x|$

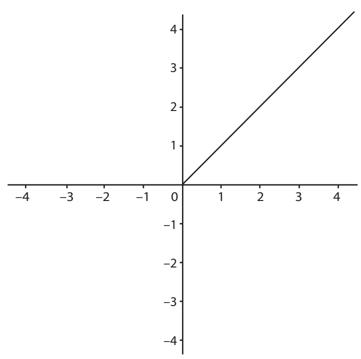

Fig. 3 $y = x, x > 0$

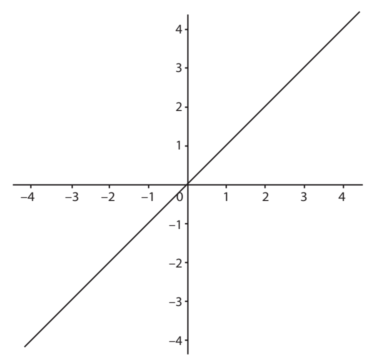

Fig. 4(a) $y = x$

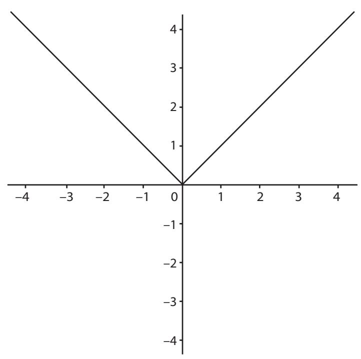

Fig. 4(b) $y = |x|$

Here are some more ‘patterns that fail’:

$3^0 = 1$	$0^3 = 0$	$\frac{0}{3} = 0$	$\frac{3}{3} = 1$	$3! = 6$
$2^0 = 1$	$0^2 = 0$	$\frac{0}{2} = 0$	$\frac{2}{2} = 1$	$2! = 2$
$1^0 = 1$	$0^1 = 0$	$\frac{0}{1} = 0$	$\frac{1}{1} = 1$	$1! = 1$
$0^0 = ?$	$0^0 = ?$	$\frac{0}{0} = ?$	$\frac{0}{0} = ?$	$0! = ?$

Perhaps you are inclined to dismiss these examples as being due to the exceptional nature of zero – we all know that zero is ‘special’ – it’s a rebel that doesn’t follow the rules. Indeed, the contradiction between the first two patterns is one way to see why 0^0 might be better left undefined. But, once you know that some patterns break down, it is far less plausible that you should believe in things because of patterns. Not all patterns continue in obvious ways (especially around zero), and what is ‘obvious’ is often a matter of opinion and experience. We cannot reliably guess mathematical structure – why should we be able to do so? And sometimes we have to sacrifice one pattern in order to satisfy another (as with defining 0! to be 1).

Problems with patterns continue even beyond number/algebra. For example, try these tempting-but-wrong ‘Copy and complete...’ statements:

An acute-angled triangle has all of its angles acute.

An obtuse-angled triangle has ________________

A right-angled triangle has ____________________

or

An equilateral triangle has 3 equal sides.

An isosceles triangle has 2 equal sides.

A scalene triangle has ______________________

or

A scalene triangle has 0 lines of symmetry.

An isosceles triangle has 1 line of symmetry.

An equilateral triangle has _________________

I think we should be cautious about suggesting that in mathematics ‘the pattern’ is a well-defined and unambiguous thing. Patterns can often be continued in different ways, and the most obvious way is not necessarily the ‘right’ one.

References

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Answers to riddles:

1. Glass
2. Bread
3. Albumen
4. Water

Note

1. Not to be confused with the famous sketch in the BBC show *The Two Ronnies*, which was a parody of the quiz show *Mastermind*, in which the contestant’s specialist subject was to “answer the question before last, each time”: see https://www.youtube.com/watch?v=aQM97rkXsHQ.

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