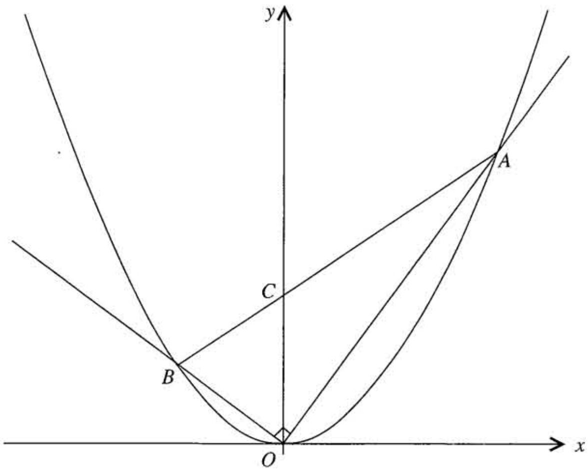
RIGHT-ANGLED TRIANGLES & PARABOLAS

Some interesting properties arise from inscribing a triangle within a parabola. Consider the parabola $y = x^2$ and the pair of straight lines $y = mx$ and $y = -\frac{x}{m}$ $(m > 0)$ which in the diagram below intersect the parabola at A and B respectively, as well as each also, of course, crossing the parabola at the origin, O. Since the lines are perpendicular (because their gradients, m and $-\frac{1}{m}$, have a product -1), angle $AOB = 90°$. Let the line AB cut the y-axis at C.



1. C is always (0, 1), regardless of the value of m.

Proof:

At A, $mx = x^2 \Rightarrow x = 0$ or m, but $x \neq 0$, since $A \neq O$.

Hence A is the point (m, m^2). Likewise, B is the point $\left(-\frac{1}{m}, \frac{1}{m^2}\right)$.

The gradient of the line AB is thus $\dfrac{m^2 - \frac{1}{m^2}}{m - \left(-\frac{1}{m}\right)} = \dfrac{\left(m + \frac{1}{m}\right)\left(m - \frac{1}{m}\right)}{m + \frac{1}{m}} = m - \frac{1}{m}$.

Hence the equation of the line AB is $\dfrac{y - m^2}{x - m} = m - \dfrac{1}{m}$ or $y - m^2 = \left(m - \frac{1}{m}\right)(x - m) = mx - \frac{x}{m} - m^2 + 1$,

which simplifies to $y = \left(m - \frac{1}{m}\right)x + 1$.

Thus C, the y-intercept, is (0, 1).

It follows from the 'angle in a semi-circle' property that for any non-vertical line through (0, 1), intersecting the parabola $y = x^2$ at points A and B, the circle on AB as diameter will pass through the origin, O. This follows because the gradient of the line AB, $m - \frac{1}{m}$, can, with suitable $m \neq 0$, take any value. We can show this by letting $m - \frac{1}{m} = r$, so $m^2 - 1 = mr$, or $m^2 - rm - 1 = 0$. This quadratic in m will have real roots if its discriminant is greater than or equal to zero. But the discriminant ("$b^2 - 4ac$" in quadratic-equation-speak) is $r^2 + 4$, which is positive for all real r, so the condition is satisfied and we can find an m corresponding to any real value of r.

2. Since ΔAOB is right-angled, it is easy to find its area:

$$\begin{aligned}
\text{Area } \Delta AOB &= \tfrac{1}{2} OA \times OB = \tfrac{1}{2}\sqrt{m^2 + (m^2)^2} \times \sqrt{\left(-\frac{1}{m}\right)^2 + \left(\frac{1}{m^2}\right)^2} \\
&= \tfrac{1}{2}\sqrt{m^2 + m^4}\sqrt{\frac{1}{m^2} + \frac{1}{m^4}} \\
&= \tfrac{1}{2}\sqrt{(m^2 + m^4)\left(\frac{m^2 + 1}{m^4}\right)} \\
&= \tfrac{1}{2}\sqrt{(1 + m^2)\left(\frac{m^2 + 1}{m^2}\right)} \\
&= \frac{m^2 + 1}{2m}
\end{aligned}$$

This neat result is consistent with what we can see: when $m = 1$ the area of the triangle is 1 square unit and this corresponds to $A = (1, 1)$, $B = (-1, 1)$, so AB is horizontal of length 2 and the y-axis is a line of symmetry. We can also see that as $m \to \infty$ the area of the triangle $\to \frac{m}{2}$. In fact the symmetrical case, with $m = 1$, gives us the *minimum* possible area.

3. We now turn to asking for what values, if any, of *m* the triangle *AOB* takes up exactly half the area enclosed by the parabola and its chord *AB*.

We'll delay the solution until next time, as it will require the use of some integration and perhaps those with some knowledge of that might like to have a shot at it. It turns out that there is essentially only one solution geometrically (allowing for reflections, etc.).

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Some interesting properties arise from inscribing a triangle within a parabola. Consider the parabola $y = x^2$ and the pair of straight lines $y = mx$ and $y = -\frac{x}{m}$ ($m > 0$), intersecting the parabola at A and B respectively, as well as each also, of course, crossing the parabola at the origin O. Since the lines are perpendicular (gradients m and $-\frac{1}{m}$ have a product of -1), angle AOB is 90°. The line AB cuts the y-axis at point C (Figure 1).

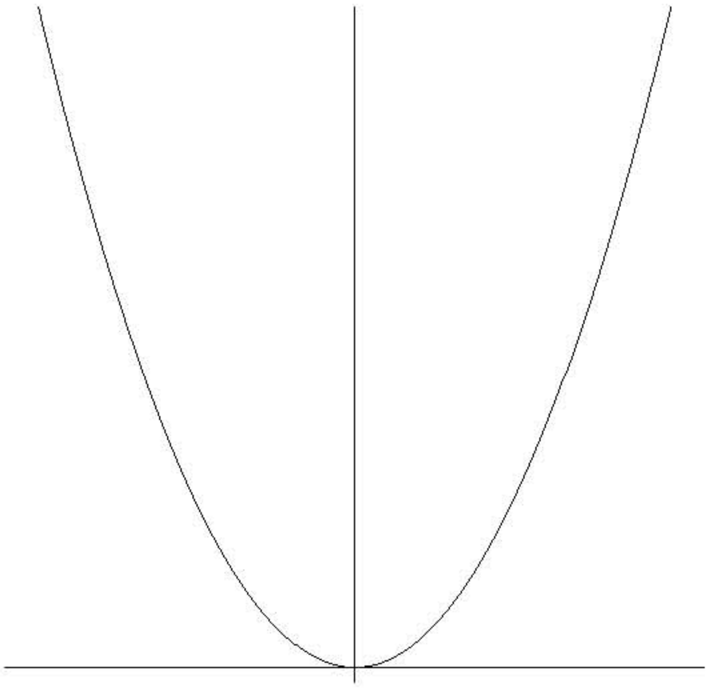

Figure 1

1. It can be shown that, regardless of the value of *m*, point *C* is always (0,1).

Proof:

The coordinates of A are found by solving the equation $mx = x^2$, giving $x = 0$ or m; hence A is $(m,\ m^2)$. Similarly, to find B, $-\frac{x}{m} = x^2$ or $mx^2 + x = 0$, so $x = -\frac{1}{m}$ or 0; hence B is $(-\frac{1}{m}, \frac{1}{m^2})$.

So the gradient of line AB is $\dfrac{m^2 - \frac{1}{m^2}}{m - -\frac{1}{m}} = \dfrac{\left(m + \frac{1}{m}\right)\left(m - \frac{1}{m}\right)}{m + \frac{1}{m}} = m - \frac{1}{m}$.

Hence the equation of line AB is $\dfrac{y - m^2}{x - m} = m - \frac{1}{m}$ or

$y - m^2 = \left(m - \frac{1}{m}\right)(x - m) = mx - \frac{x}{m} - m^2 + 1$, so $y = \left(m - \frac{1}{m}\right)x + 1$.

Hence, point C (the y-intercept) is (0,1).

It follows from the angle in a semicircle property that for any non-vertical line passing through (0,1) and intersecting the parabola $y = x^2$ at A and B, if a circle is constructed with the line segment AB as diameter, then this circle passes through the origin O. This follows because the gradient of the line AB, $m-\frac{1}{m}$, can, with suitable $m \neq 0$, take any value. We can show this by writing $m-\frac{1}{m}=r$ so $m^2-1=mr$ or $m^2-rm-1=0$. Since this is a quadratic in m, m will be real if and only if the discriminant is greater than or equal to zero. So the condition is $r^2+4\geq 0$, but since this is always satisfied, for all real r, then for any value of r there exists a real m, so $m-\frac{1}{m}$ can be made to take any real value.

2. Since ΔAOB is right-angled, it is easy to find its area:

$$Area\ \Delta AOB = \tfrac{1}{2}OA \times OB$$
$$= \tfrac{1}{2}\sqrt{m^2+(m^2)^2}\times\sqrt{\left(-\frac{1}{m}\right)^2+\left(\frac{1}{m^2}\right)^2}$$
$$= \tfrac{1}{2}\sqrt{m^2+m^4}\sqrt{\frac{1}{m^2}+\frac{1}{m^4}}$$
$$= \tfrac{1}{2}\sqrt{(m^2+m^4)\left(\frac{1}{m^2}+\frac{1}{m^4}\right)}$$
$$= \tfrac{1}{2}\sqrt{1+m^2+\frac{1}{m^2}+1}$$
$$= \tfrac{1}{2}\sqrt{m^2+2+\frac{1}{m^2}}$$
$$= \tfrac{1}{2}\sqrt{\frac{m^4+2m^2+1}{m^2}}$$
$$= \frac{1}{2m}\sqrt{(m^2+1)^2}$$
$$= \frac{m^2+1}{2m}$$

This neat result is consistent with what we can see; for instance, that when $m = 1$ the area of the triangle is 1 square unit: this is the case with a vertical line of symmetry, where AB is horizontal, A is (1, 1) and B is (–1, 1). We can also see that as $m \to \infty$ the area of the triangle $\to \frac{m}{2}$.

3. This leads us to ask for what value(s) of *m* will the area of the triangle *AOB* be exactly *half* of the area enclosed by the curve and the line *AB*. This problem turns out to have a unique solution.

The condition is equivalent to saying that the area between OB and the curve plus the area between OA and the curve is equal to the area of triangle AOB.

Hence,

$$\int_{-\frac{1}{m}}^{0}\left(-\frac{x}{m}-x^2\right)dx+\int_{0}^{m}\left(mx-x^2\right)dx=\frac{m^2+1}{2m}$$

$$\int_{0}^{-\frac{1}{m}}\left(\frac{x}{m}+x^2\right)dx+\left[\frac{mx^2}{2}-\frac{x^3}{3}\right]_0^m=\frac{m^2+1}{2m}$$

$$\left[\frac{x^2}{2m}+\frac{x^3}{3}\right]_0^{-\frac{1}{m}}+\frac{m^3}{2}-\frac{m^3}{3}=\frac{m^2+1}{2m}$$

$$\frac{1}{2m^3}-\frac{1}{3m^3}+\frac{m^3}{6}=\frac{m^2+1}{2m}$$

$$\frac{1}{6m^3}+\frac{m^3}{6}=\frac{m^2+1}{2m}$$

$$1+m^6=3m^2(m^2+1)$$

which, making the substitution $p=m^2$, is a cubic, $p^3-3p^2-3p+1=0$, which factorises to give $(p+1)(p^2-4p+1)=0$, giving $p=-1$ (impossible, since $p=m^2$) or $p=2\pm\sqrt{3}$. Hence, $m=\pm\sqrt{2+\sqrt{3}}$ or $m=\pm\sqrt{2-\sqrt{3}}$.

However, these four solutions turn out to be just one solution geometrically. This can be seen as follows. Writing, for example, $\sqrt{2+\sqrt{3}} = \sqrt{a}+\sqrt{b}$ and squaring both sides gives $2+\sqrt{3}=a+b+2\sqrt{ab}$, so $a+b=2$ and $4ab=3$, and solving these simultaneous equations we obtain $a=\frac{3}{2}$ and $b=\frac{1}{2}$, so $\sqrt{2+\sqrt{3}}=\frac{\sqrt{6}}{2}+\frac{\sqrt{2}}{2}$. Similarly, we find that $\sqrt{2-\sqrt{3}}=\frac{\sqrt{6}}{2}-\frac{\sqrt{2}}{2}$. The other two solutions ($m=-\sqrt{2+\sqrt{3}}=-\frac{\sqrt{6}}{2}-\frac{\sqrt{2}}{2}$ and $m=-\sqrt{2-\sqrt{3}}=-\frac{\sqrt{6}}{2}+\frac{\sqrt{2}}{2}$) are just reflections in the *y*-axis of the lines resulting from the two solutions obtained above, and therefore represent the same geometrical situations. It only remains to show that the solutions $m_1=\sqrt{2+\sqrt{3}}=\frac{\sqrt{6}}{2}+\frac{\sqrt{2}}{2}$ and $m_2=\sqrt{2-\sqrt{3}}=\frac{\sqrt{6}}{2}-\frac{\sqrt{2}}{2}$ are also descriptions of the same geometry as each other, and this is readily done by simplifying

$$-\frac{1}{m_1}=-\frac{1}{\sqrt{2+\sqrt{3}}}=-\frac{1}{\left(\frac{\sqrt{6}}{2}+\frac{\sqrt{2}}{2}\right)}=-\frac{2}{\sqrt{6}+\sqrt{2}}=-\frac{2(\sqrt{6}-\sqrt{2})}{(\sqrt{6}+\sqrt{2})(\sqrt{6}-\sqrt{2})}=\frac{-2(\sqrt{6}-\sqrt{2})}{6-2}$$

$=\frac{\sqrt{2}-\sqrt{6}}{2}=-\sqrt{2-\sqrt{3}}=-m_2$, which was one of the other solutions. Hence, we have a unique solution to the problem.

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