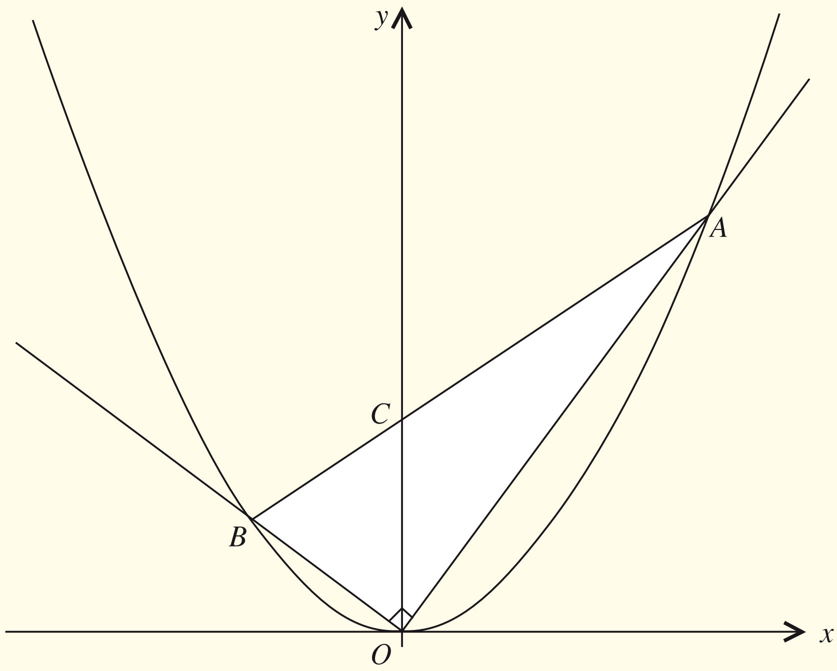
RIGHT-ANGLED TRIANGLES & PARABOLAS

Some interesting properties arise from inscribing a triangle within a parabola. Consider the parabola $y = x^2$ and the pair of straight lines $y = mx$ and $y = -\frac{x}{m}$ $(m > 0)$ which in the diagram below intersect the parabola at A and B respectively, as well as each also, of course, crossing the parabola at the origin, O. Since the lines are perpendicular (because their gradients, m and $-\frac{1}{m}$, have a product -1), angle $AOB = 90°$. Let the line AB cut the y-axis at C.



1. *C* is always (0, 1), regardless of the value of *m*.

<u>*Proof*</u>:

At A, $mx = x^2 \Rightarrow x = 0$ or m, but $x \neq 0$, since $A \neq$ O.

Hence A is the point (m, m^2). Likewise, B is the point $\left(-\frac{1}{m}, \frac{1}{m^2}\right)$.

The gradient of the line AB is thus $\dfrac{m^2 - \frac{1}{m^2}}{m - \left(-\frac{1}{m}\right)} = \dfrac{\left(m + \frac{1}{m}\right)\left(m - \frac{1}{m}\right)}{m + \frac{1}{m}} = m - \frac{1}{m}$.

Hence the equation of the line AB is $\dfrac{y - m^2}{x - m} = m - \frac{1}{m}$ or $y - m^2 = \left(m - \frac{1}{m}\right)(x - m) = mx - \frac{x}{m} - m^2 + 1$,

which simplifies to $y = \left(m - \frac{1}{m}\right)x + 1$.

Thus C, the y-intercept, is (0, 1).

It follows from the 'angle in a semi-circle' property that for any non-vertical line through (0, 1), intersecting the parabola $y = x^2$ at points A and B, the circle on AB as diameter will pass through the origin, O. This follows because the gradient of the line AB, $m - \frac{1}{m}$, can, with suitable $m \neq 0$, take any value. We can show this by letting $m - \frac{1}{m} = r$, so $m^2 - 1 = mr$, or $m^2 - rm - 1 = 0$. This quadratic in m will have real roots if its discriminant is greater than or equal to zero. But the discriminant ("$b^2 - 4ac$" in quadratic-equation-speak) is $r^2 + 4$, which is positive for all real r, so the condition is satisfied and we can find an m corresponding to any real value of r.

2. Since ΔAOB is right-angled, it is easy to find its area:

$$\begin{aligned}\text{Area } \Delta AOB &= \tfrac{1}{2}OA\times OB = \tfrac{1}{2}\sqrt{m^2+(m^2)^2}\times\sqrt{\left(-\frac{1}{m}\right)^2+\left(\frac{1}{m^2}\right)^2}\\ &= \tfrac{1}{2}\sqrt{m^2+m^4}\sqrt{\frac{1}{m^2}+\frac{1}{m^4}}\\ &= \tfrac{1}{2}\sqrt{(m^2+m^4)\left(\frac{m^2+1}{m^4}\right)}\\ &= \tfrac{1}{2}\sqrt{(1+m^2)\left(\frac{m^2+1}{m^2}\right)}\\ &= \frac{m^2+1}{2m}\end{aligned}$$

This neat result is consistent with what we can see: when $m = 1$ the area of the triangle is 1 square unit and this corresponds to $A = (1, 1)$, $B = (-1, 1)$, so AB is horizontal of length 2 and the y-axis is a line of symmetry. We can also see that as $m \to \infty$ the area of the triangle $\to \dfrac{m}{2}$. In fact the symmetrical case, with $m = 1$, gives us the *minimum* possible area.

3. We now turn to asking for what values, if any, of *m* the triangle *AOB* takes up exactly half the area enclosed by the parabola and its chord *AB*.

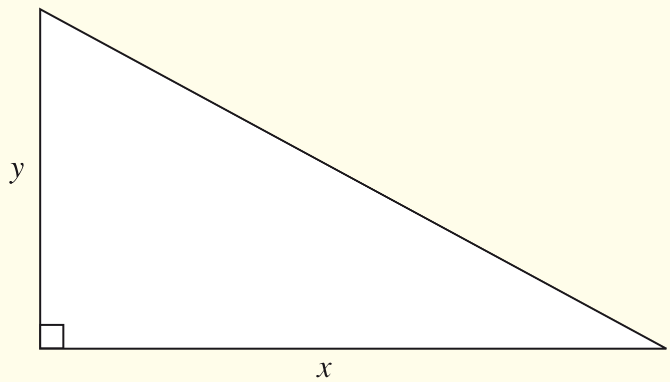
We'll delay the solution until next time, as it will require the use of some integration and perhaps those with some knowledge of that might like to have a shot at it. It turns out that there is essentially only one solution geometrically (allowing for reflections, etc.).

Colin Foster

AREA & PERIMETER

In our editorial for Issue 30, we casually mentioned that 30 appeared as the common value for perimeter and area in one of the only two Pythagorean triangles (i.e. right-angled with integer sides) in which these two numbers are the same. One, as we said then, is the 5-12-13 triangle. What is the other?

Here we have a right-angled triangle with sides enclosing the right angle of lengths x and y. The hypotenuse will be of length $\sqrt{x^2+y^2}$, so the two values we are interested in are $P = x + y + \sqrt{x^2+y^2}$ and $A = \tfrac{1}{2}xy$.



Equating these and rearranging we have

$$\sqrt{x^2+y^2} = \tfrac{1}{2}xy - (x+y)$$

Hence $\quad x^2+y^2 = \tfrac{1}{4}x^2y^2 - x^2y - xy^2 + x^2 + 2xy + y^2$

We can simplify this to give $x^2y^2 - 4x^2y - 4xy^2 + 8xy = 0$, and we notice that this expression has xy as a factor. This cannot possibly be zero, so we can cancel it to give a rather simpler equation: $xy - 4x - 4y + 8 = 0$. Now for the cunning bit. If we add 8 to both sides the new LHS will factorise: $(x - 4)(y - 4) = 8 = 1 \times 8$ or 2×4. Since x and y are positive integers, we must be able to equate the factors and there are only two possibilities. From the first we have $(x, y) = (5, 12)$ and from the second we have $(x, y) = (6, 8)$. The first gives us the triangle we know about, 5-12-13, and the second gives us the only other possibility, 6-8-10, for which both perimeter and area (in the appropriate units) are 24.

MLP