Some interesting properties arise from inscribing a triangle within a parabola. Consider the parabola \( y = x^2 \) and the pair of straight lines \( y = mx \) and \( y = -\frac{x}{m} \) (\( m > 0 \)) which in the diagram below intersect the parabola at \( A \) and \( B \) respectively, as well as each also, of course, crossing the parabola at the origin, \( O \). Since the lines are perpendicular (because their gradients, \( m \) and \( -\frac{1}{m} \), have a product –1), angle \( AOB = 90^\circ \). Let the line \( AB \) cut the \( y \)-axis at \( C \).

1. \( C \) is always \((0, 1)\), regardless of the value of \( m \).

*Proof:*

At \( A \), \( mx = x^2 \Rightarrow x = 0 \) or \( m \), but \( x \neq 0 \), since \( A \neq O \).

Hence \( A \) is the point \((m, m^2)\). Likewise, \( B \) is the point \((-\frac{1}{m}, 1)\).

The gradient of the line \( AB \) is thus

\[
\frac{m^2 - \frac{1}{m^2}}{m - \left(-\frac{1}{m}\right)} = \frac{(m + \frac{1}{m})(m - \frac{1}{m})}{m + \frac{1}{m}} = m - \frac{1}{m}.
\]

Hence the equation of the line \( AB \) is

\[
y - m^2 = m - \frac{1}{m} \text{ or } y - m^2 = \left(m - \frac{1}{m}\right)(x - m) = mx - \frac{x - m^2}{m} + 1,
\]

which simplifies to \( y = \left(m - \frac{1}{m}\right)x + 1 \).

Thus \( C \), the \( y \)-intercept, is \((0, 1)\).

It follows from the ‘angle in a semi-circle’ property that for any non-vertical line through \((0, 1)\), intersecting the parabola \( y = x^2 \) at points \( A \) and \( B \), the circle on \( AB \) as diameter will pass through the origin, \( O \). This follows because the gradient of the line \( AB \), \( m - \frac{1}{m} \), can, with suitable \( m \neq 0 \), take any value. We can show this by letting \( m - \frac{1}{m} = r \), so \( m^2 - 1 = mr \), or \( m^2 - rm - 1 = 0 \). This quadratic in \( m \) will have real roots if its discriminant is greater than or equal to zero. But the discriminant ("\( b^2 - 4ac \)" in quadratic-equation-speak) is \( r^2 + 4 \), which is positive for all real \( r \), so the condition is satisfied and we can find an \( m \) corresponding to any real value of \( r \).
2. Since $\Delta AOB$ is right-angled, it is easy to find its area:

\[
\text{Area } \Delta AOB = \frac{1}{2} OA \times OB = \frac{1}{2} \sqrt{m^2 + (m^2)^2} \times \sqrt{\left(\frac{1}{m}\right)^2 + \left(\frac{1}{m^2}\right)^2}
\]

\[
= \frac{1}{2} \sqrt{m^2 + m^4} \sqrt{\frac{1}{m^2} + \frac{1}{m^4}}
\]

\[
= \frac{1}{2} \sqrt{(m^2 + m^4) \left(\frac{m^2 + 1}{m^4}\right)}
\]

\[
= \frac{1}{2} \sqrt{(1 + m^2) \left(\frac{m^2 + 1}{m^2}\right)}
\]

\[
= \frac{m^2 + 1}{2m}
\]

This neat result is consistent with what we can see: when $m = 1$ the area of the triangle is 1 square unit and this corresponds to $A = (1, 1), B = (-1, 1)$, so $AB$ is horizontal of length 2 and the $y$-axis is a line of symmetry. We can also see that as $m \to \infty$ the area of the triangle $\to \frac{m}{2}$. In fact the symmetrical case, with $m = 1$, gives us the minimum possible area.

3. We now turn to asking for what values, if any, of $m$ the triangle $AOB$ takes up exactly half the area enclosed by the parabola and its chord $AB$.

We’ll delay the solution until next time, as it will require the use of some integration and perhaps those with some knowledge of that might like to have a shot at it. It turns out that there is essentially only one solution geometrically (allowing for reflections, etc.).

Colin Foster

**AREA & PERIMETER**

In our editorial for Issue 30, we casually mentioned that 30 appeared as the common value for perimeter and area in one of the only two Pythagorean triangles (i.e. right-angled with integer sides) in which these two numbers are the same. One, as we said then, is the 5-12-13 triangle. What is the other?

Here we have a right-angled triangle with sides enclosing the right angle of lengths $x$ and $y$. The hypotenuse will be of length $\sqrt{x^2 + y^2}$, so the two values we are interested in are $P = x + y + \sqrt{x^2 + y^2}$ and $A = \frac{1}{2}xy$.

Equating these and rearranging we have

\[
\sqrt{x^2 + y^2} = \frac{1}{2} xy - (x + y)
\]

Hence

\[
x^2 + y^2 = \frac{1}{4}x^2y^2 - x^2y - xy^2 + x^2 + 2xy + y^2
\]

We can simplify this to give $x^2y^2 - 4x^2y - 4xy^2 + 8xy = 0$, and we notice that this expression has $xy$ as a factor. This cannot possibly be zero, so we can cancel it to give a rather simpler equation: $xy - 4x - 4y + 8 = 0$. Now for the cunning bit. If we add 8 to both sides the new LHS will factorise: $(x - 4)(y - 4) = 8 = 1 \times 8$ or $2 \times 4$. Since $x$ and $y$ are positive integers, we must be able to equate the factors and there are only two possibilities. From the first we have $(x, y) = (5, 12)$ and from the second we have $(x, y) = (6, 8)$. The first gives us the triangle we know about, 5-12-13, and the second gives us the only other possibility, 6-8-10, for which both perimeter and area (in the appropriate units) are 24.