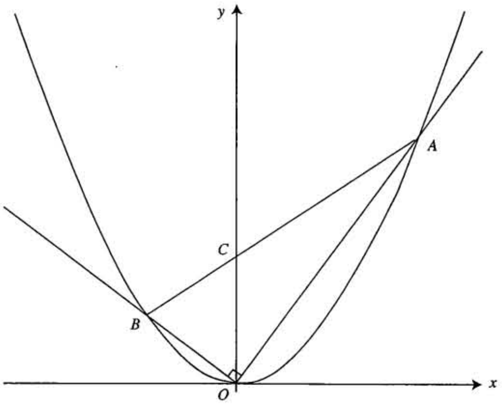
TRIANGLES AND PARABOLAS ~ the hard part!

We left you with a problem last time – for those who know about finding areas under curves, or between curves and straight lines. Remember that *OA* has slope *m* and *OB* is at right angles to it. For what values, if any, of *m* does the triangle *AOB* take up half the area which *AB* cuts off from the parabola $y = x^2$?



This is equivalent to saying that the area between *OB* and the curve plus the area between *OA* and the curve is equal to the area of the triangle *AOB* which we calculated last time as being $\frac{m^2+1}{2m}$. $A=(m,m^2)$; $B=\left(-\frac{1}{m},\frac{1}{m^2}\right)$.

Hence we have the following:

$$\int_{-\frac{1}{m}}^{0}\left(-\frac{x}{m}-x^2\right)\mathrm{d}x+\int_0^m (mx-x^2)\,\mathrm{d}x=\frac{m^2+1}{2m}$$

$$\therefore \quad \left[-\frac{x^2}{2m}-\frac{x^3}{3}\right]_{-\frac{1}{m}}^{0}+\left[\frac{mx^2}{2}-\frac{x^3}{3}\right]_0^m=\frac{m^2+1}{2m}$$

$$\therefore \quad \frac{1}{2m^3}-\frac{1}{3m^3}+\frac{m^3}{2}-\frac{m^3}{3}=\frac{m^2+1}{2m}$$

Multiply by $6m^3$ and collect up terms:

$$1+m^6=3m^2(m^2+1)$$

We now make the substitution $p=m^2$ to create the cubic equation $p^3-3p^2-3p+1=0$.

This factorises to give $(p+1)(p^2-4p+1)=0$, giving $p=-1$ (impossible, as $p=m^2$) or $p=2\pm\sqrt{3}$.

Hence $m=\pm\sqrt{2+\sqrt{3}}$ or $m=\pm\sqrt{2-\sqrt{3}}$. But we claimed last time that there was essentially one solution geometrically, so how can we prove that?

If we multiply the positive value of one possibility, by the negative value of the other we get $-\sqrt{(2+\sqrt{3})(2-\sqrt{3})}$, but this simplifies to -1, so that if one value of *m* corresponds to the line *OA*, the appropriate other value gives *OB*, and vice versa. The other values give the mirror images in the *y*-axis, so essentially there is, geometrically, a unique solution.

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Right-Angled Triangles and Parabolas

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Some interesting properties arise from inscribing a triangle within a parabola. Consider the parabola $y = x^2$ and the pair of straight lines $y = mx$ and $y = -\frac{x}{m}$ ($m > 0$), intersecting the parabola at *A* and *B* respectively, as well as each also, of course, crossing the parabola at the origin *O*. Since the lines are perpendicular (gradients *m* and $-\frac{1}{m}$ have a product of –1), angle *AOB* is 90°. The line *AB* cuts the *y*-axis at point *C* (Figure 1).

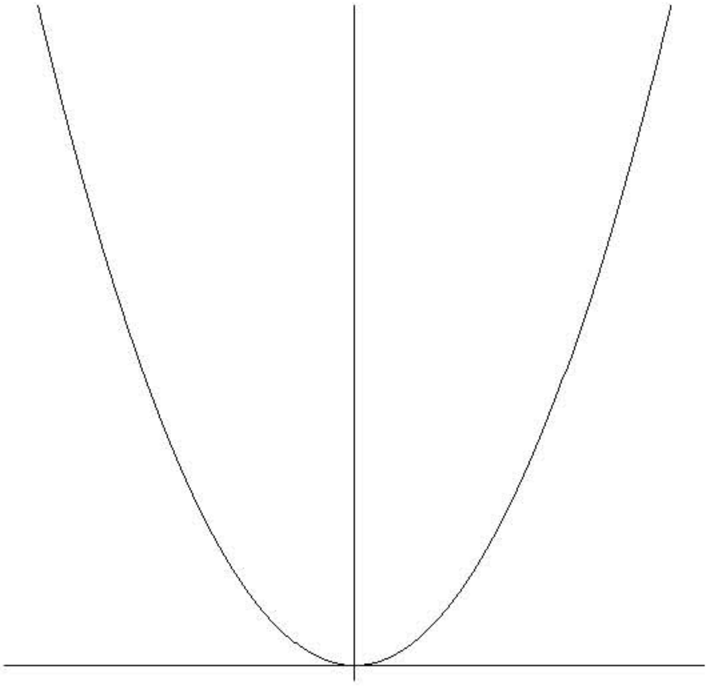

Figure 1

1. It can be shown that, regardless of the value of *m*, point *C* is always (0,1).

Proof:

The coordinates of *A* are found by solving the equation $mx = x^2$, giving $x = 0$ or m; hence *A* is $(m,\ m^2)$. Similarly, to find B, $-\frac{x}{m} = x^2$ or $mx^2 + x = 0$, so $x = -\frac{1}{m}$ or 0; hence *B* is $(-\frac{1}{m}, \frac{1}{m^2})$.

So the gradient of line *AB* is $\dfrac{m^2 - \frac{1}{m^2}}{m - -\frac{1}{m}} = \dfrac{\left(m + \frac{1}{m}\right)\left(m - \frac{1}{m}\right)}{m + \frac{1}{m}} = m - \frac{1}{m}$.

Hence the equation of line *AB* is $\dfrac{y - m^2}{x - m} = m - \frac{1}{m}$ or

$y - m^2 = \left(m - \frac{1}{m}\right)(x - m) = mx - \frac{x}{m} - m^2 + 1$, so $y = \left(m - \frac{1}{m}\right)x + 1$.

Hence, point C (the y-intercept) is (0,1).

It follows from the angle in a semicircle property that for any non-vertical line passing through (0,1) and intersecting the parabola $y = x^2$ at A and B, if a circle is constructed with the line segment AB as diameter, then this circle passes through the origin O. This follows because the gradient of the line AB, $m - \frac{1}{m}$, can, with suitable $m \neq 0$, take any value. We can show this by writing $m - \frac{1}{m} = r$ so $m^2 - 1 = mr$ or $m^2 - rm - 1 = 0$. Since this is a quadratic in m, m will be real if and only if the discriminant is greater than or equal to zero. So the condition is $r^2 + 4 \geq 0$, but since this is always satisfied, for all real r, then for any value of r there exists a real m, so $m - \frac{1}{m}$ can be made to take any real value.

2. Since ΔAOB is right-angled, it is easy to find its area:

$$\begin{aligned}
Area\ \Delta AOB &= \tfrac{1}{2} OA \times OB \\
&= \tfrac{1}{2}\sqrt{m^2 + (m^2)^2} \times \sqrt{\left(-\frac{1}{m}\right)^2 + \left(\frac{1}{m^2}\right)^2} \\
&= \tfrac{1}{2}\sqrt{m^2 + m^4}\sqrt{\frac{1}{m^2} + \frac{1}{m^4}} \\
&= \tfrac{1}{2}\sqrt{(m^2 + m^4)\left(\frac{1}{m^2} + \frac{1}{m^4}\right)} \\
&= \tfrac{1}{2}\sqrt{1 + m^2 + \frac{1}{m^2} + 1} \\
&= \tfrac{1}{2}\sqrt{m^2 + 2 + \frac{1}{m^2}} \\
&= \tfrac{1}{2}\sqrt{\frac{m^4 + 2m^2 + 1}{m^2}} \\
&= \frac{1}{2m}\sqrt{(m^2 + 1)^2} \\
&= \frac{m^2 + 1}{2m}
\end{aligned}$$

This neat result is consistent with what we can see; for instance, that when $m = 1$ the area of the triangle is 1 square unit: this is the case with a vertical line of symmetry, where AB is horizontal, A is (1, 1) and B is (–1, 1). We can also see that as $m \to \infty$ the area of the triangle $\to \frac{m}{2}$.

3. This leads us to ask for what value(s) of *m* will the area of the triangle *AOB* be exactly *half* of the area enclosed by the curve and the line *AB*. This problem turns out to have a unique solution.

The condition is equivalent to saying that the area between OB and the curve plus the area between OA and the curve is equal to the area of triangle AOB.

Hence,

$$\int_{-\frac{1}{m}}^{0}\left(-\frac{x}{m}-x^2\right)dx+\int_0^m\left(mx-x^2\right)dx=\frac{m^2+1}{2m}$$

$$\int_0^{-\frac{1}{m}}\left(\frac{x}{m}+x^2\right)dx+\left[\frac{mx^2}{2}-\frac{x^3}{3}\right]_0^m=\frac{m^2+1}{2m}$$

$$\left[\frac{x^2}{2m}+\frac{x^3}{3}\right]_0^{-\frac{1}{m}}+\frac{m^3}{2}-\frac{m^3}{3}=\frac{m^2+1}{2m}$$

$$\frac{1}{2m^3}-\frac{1}{3m^3}+\frac{m^3}{6}=\frac{m^2+1}{2m}$$

$$\frac{1}{6m^3}+\frac{m^3}{6}=\frac{m^2+1}{2m}$$

$$1+m^6=3m^2(m^2+1)$$

which, making the substitution $p=m^2$, is a cubic, $p^3-3p^2-3p+1=0$, which factorises to give
$(p+1)(p^2-4p+1)=0$, giving $p=-1$ (impossible, since $p=m^2$) or $p=2\pm\sqrt{3}$.
Hence, $m=\pm\sqrt{2+\sqrt{3}}$ or $m=\pm\sqrt{2-\sqrt{3}}$.

However, these four solutions turn out to be just one solution geometrically. This can be seen as follows. Writing, for example, $\sqrt{2+\sqrt{3}}=\sqrt{a}+\sqrt{b}$ and squaring both sides gives $2+\sqrt{3}=a+b+2\sqrt{ab}$, so $a+b=2$ and $4ab=3$, and solving these simultaneous equations we obtain $a=\frac{3}{2}$ and $b=\frac{1}{2}$, so $\sqrt{2+\sqrt{3}}=\frac{\sqrt{6}}{2}+\frac{\sqrt{2}}{2}$. Similarly, we find that $\sqrt{2-\sqrt{3}}=\frac{\sqrt{6}}{2}-\frac{\sqrt{2}}{2}$. The other two solutions ($m=-\sqrt{2+\sqrt{3}}=-\frac{\sqrt{6}}{2}-\frac{\sqrt{2}}{2}$ and $m=-\sqrt{2-\sqrt{3}}=-\frac{\sqrt{6}}{2}+\frac{\sqrt{2}}{2}$) are just reflections in the *y*-axis of the lines resulting from the two solutions obtained above, and therefore represent the same geometrical situations. It only remains to show that the solutions $m_1=\sqrt{2+\sqrt{3}}=\frac{\sqrt{6}}{2}+\frac{\sqrt{2}}{2}$ and $m_2=\sqrt{2-\sqrt{3}}=\frac{\sqrt{6}}{2}-\frac{\sqrt{2}}{2}$ are also descriptions of the same geometry as each other, and this is readily done by simplifying

$$-\frac{1}{m_1}=-\frac{1}{\sqrt{2+\sqrt{3}}}=-\frac{1}{\left(\frac{\sqrt{6}}{2}+\frac{\sqrt{2}}{2}\right)}=-\frac{2}{\sqrt{6}+\sqrt{2}}=-\frac{2(\sqrt{6}-\sqrt{2})}{(\sqrt{6}+\sqrt{2})(\sqrt{6}-\sqrt{2})}=\frac{-2(\sqrt{6}-\sqrt{2})}{6-2}$$

$=\frac{\sqrt{2}-\sqrt{6}}{2}=-\sqrt{2-\sqrt{3}}=-m_2$, which was one of the other solutions. Hence, we have a unique solution to the problem.

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