Lesson plan: MATHS KS4

COUNTING OUT

A counting-out game provides a context for students to apply knowledge of sequences, multiples and powers, says Colin Foster

WHY TEACH THIS?

This lesson invites students to make sense of a counting-out game to predict who will be the winner.

KEY CURRICULUM LINKS

- Make and test conjectures about the generalisations that underlie patterns and relationships; look for proofs or counter-examples; begin to use algebra to support and construct arguments
- Select appropriate concepts, methods and techniques to apply to unfamiliar and non-routine problems

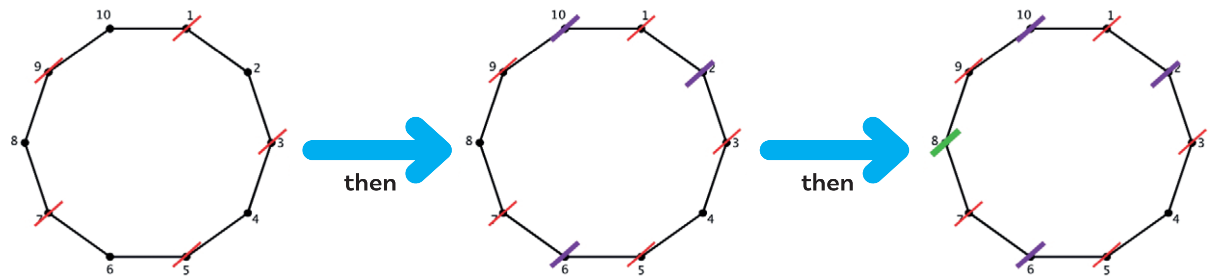
In this lesson, students analyse a counting-out game, known as the *Josephus problem*, in which people are excluded one at a time until the person left at the end is the winner. Students explore which person will be the winner with different numbers of starting people and find connections with arithmetic sequences and powers of 2.

Q In a counting-out game, how can you predict who will win?

STARTER ACTIVITY

Q *Imagine 10 people sitting in a circle numbered 1 to 10 in order. You go round the circle counting 'out' every other person, starting with person 1, until there is just one person left. Who will be left?*

Students may get different answers depending on how they interpret the instructions. They need to completely ignore anyone who is already out. Go through this together until everyone agrees that the answer is person 4.



red crossings out starting with person 1
1, 3, 5, 7, 9

purple crossings out starting with person 2
2, 6, 10

green crossings out starting with person 8
8

MAIN ACTIVITY

Q *Why should it be person 4 who wins? How does the 4 depend on the total number of people? Try this counting-out game with different numbers of people and see what happens.*

Students may come up with wild ideas about possible connections between 10 and 4 (subtract 6, halve it and add 1, etc.). This is fine, as these will be contradicted very soon as students gather more data and gain more insight into the problem.

Generating data systematically reveals the key role of powers of 2. Whenever there are 2^m people, the winner is always the 2^mth person. Students could explore what happens when the number of people is, say, one more or one fewer than this.

Students could start by acting out the process with real students sitting in a circle. A tap on the shoulder means that they are 'out'. If they collect their results in a table, they will notice that the pattern in the answers seems to be *multiples* of 2, which resets every time we hit a *power* of 2:

Total number of people	*Last person standing*	*Patterns noticed*
1	1	2^0
2	2	2^1
3	2	2
4	4	2^2
5	2	2
6	4	4
7	6	6
8	8	2^3
9	2	2
10	4	4
11	6	6
12	8	8
13	10	10
14	12	12
15	14	14
16	16	2^4

DISCUSSION

Q *What did you find out? How does who wins depend on how many people there are at the start?*

Students will notice things like the fact that the odd numbers get eliminated first, and then alternate even numbers, and so on. So the answer will never be an odd-numbered person, except for the case where there is just one person to start with.

It is hard to obtain a formula for the winning person, but students may be able to figure out who the winner is, even for quite large numbers, without having to write out all the numbers. For example, you could challenge them to work out which would be the winning number for all the students in the class, without actually doing it.

Supposing that there are around 30 students in the class, students could reason as follows:

- For 32 students (a power of 2), the winner will be student 32.
- For 33 students, it will be student 2.
- For 34 students, it will be student 4.

...and so on.

Similarly, for 31 students, we will need to calculate 32 – 2 = 30, so it will be student 30. For 30 students, it will be student 32 – 2 – 2 = 28, and so on. Getting a full formula is likely to be too hard, but students may get some way towards expressing this.

Let n be the total number of people. Then winner = $2(n - p)$, where p is the highest power of 2 less than n.

If we want to express p in terms of n as well, we can do so using *logarithms* and the *ceiling function* (indicated with $\lceil x \rceil$ brackets, which means the closest integer greater than or equal to x). Using this notation,

$$p = 2^{\lceil \log_2 n \rceil - 1}$$

so the winner is given by

$$\text{winner} = 2\left(n - 2^{\lceil \log_2 n \rceil - 1}\right)$$

A spreadsheet of values is available at **teachwire.net/counting-winners**

Free Mathematics Support Resources from the Association of Teachers of Mathematics

During this uncertain time, while our normal teaching pattern has been disrupted, ATM has made available a selection of Free to Access resources suitable for use within schools and home-learning environments. We will be adding to these resources in the weeks ahead. **atm.org.uk/Free-ATM-Resources-**

ADDITIONAL RESOURCE

An interactive version of the problem (but starting by deleting person 2, rather than person 1) is at **bit.ly/geo-josephus**

GOING DEEPER

Confident students could explore what happens if you miss out ***two*** people, instead of one person,

ABOUT OUR EXPERT

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