KS4 Maths Lesson Plan – Quadratic Equations Bring Together A Host Of Mathematical Ideas And Techniques.

Finding quadratic expressions that satisfy a given constraint leads to useful exploration of factorisation, says Colin Foster

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Quadratic expressions are considerably more complicated to work with than linear expressions, and pupils often find them hard to handle.

Students may carry out factorising of quadratics by applying poorly-understood procedures that make little mathematical sense to them, and this can be especially so when it comes to non-monic quadratics (those where the coefficient of x^2 is not 1).

In this lesson, students approach factorising non-monic quadratics by trying to find factorisations which will expand and simplify to produce a quadratic expression of a specified form.

This entails lots of useful practice at expanding pairs of brackets and collecting like terms, and also gives opportunity for students to unpick what is going on, so as to gain insight into how the inverse process of factorising works.

By working backwards in this way to obtain the necessary factors, students build a deeper understanding of factorising quadratics.

Starter activity

Q. Expand and simplify these expressions:

$(2x + 1)(3x - 4)$
$(2x - 1)(3x + 4)$
$(2x + 1)(3x + 4)$
$(2x - 1)(3x - 4)$
$(3x + 1)(2x - 4)$
$(3x - 1)(2x + 4)$
$(3x + 1)(2x + 4)$
$(3x - 1)(2x - 4)$

What patterns do you notice in the questions and the answers? Can you explain why they happen?

(A resource sheet containing all of the tasks in today's lesson is available here.)

The patterns in the questions cover all of the possible pairings of $2x/3x$ with $\pm 1/\pm 4$, so there are 8 possibilities altogether.

The expanded and simplified expressions are:

$(2x + 1)(3x - 4) = 6x^2 - 5x - 4$
$(2x - 1)(3x + 4) = 6x^2 + 5x - 4$
$(2x + 1)(3x + 4) = 6x^2 + 11x + 4$
$(2x - 1)(3x - 4) = 6x^2 - 11x + 4$
$(3x + 1)(2x - 4) = 6x^2 - 10x - 4$
$(3x - 1)(2x + 4) = 6x^2 + 10x - 4$
$(3x + 1)(2x + 4) = 6x^2 + 14x + 4$
$(3x - 1)(2x - 4) = 6x^2 - 14x + 4$

Students may start by making observations about the factorised versions, noting the $2x/3x$ and $\pm1/\pm4$ variation. If so, encourage them also to focus on the expanded-and-simplified expressions. Students may notice that the first term is always $6x^2$ and that the constant term is always ±4. They should be able to explain why this is, so it could be helpful to ask:

Q. Why does that happen? Where does that part come from? Why is it positive/negative?

Students may also make conjectures about the values (and sign) of the middle term. That is much harder to rationalise fully without carrying out the expanding and simplifying, and students will look more deeply at this aspect in the main part of the lesson.

It may be necessary at this point to review the process of expanding brackets and simplifying the result, and to remind students how to handle directed numbers. It can be a demanding process, because students need to be competent with both multiplication and addition/subtraction of directed numbers (at the same time!).

For example, for $(2x + 1)(3x - 4)$, students need to do things like $2 \times (-4x) = -8x$, but then also $3x - 8x = -5x$. So there is lots of opportunity for errors and confusion.

It is not necessary for everyone to be completely competent at this before the main activity, as they will get lots of practice at the procedure later in the lesson. Laying things out in a grid may help:

×	$3x$	-4
$2x$	$6x^2$	$-8x$
$+1$	$3x$	-4

$(2x + 1)(3x - 4) = 6x^2 - 5x - 4$

Main activity

Q. Now we are going to work backwards from some expanded expressions that have patterns in to see if we can factorise them back into brackets.

Here is a quadratic expression in which the constant term is missing:

$$6x^2 + 7x + \square$$

Let's say that the number in the box must be an integer between –10 and 10.

Can we choose a number for the box that will make the expression factorisable?

If students are stuck with this starting point, you could suggest that they try putting 2 in the box to start with, which leads to a factorisable expression:

$$6x^2 + 7x + 2 = (2x + 1)(3x + 2)$$

Then you could ask:

Q. Can you find any other numbers that will work?

Students could either try other numbers in the range –10 to 10, or begin with the factorised forms and aim to create the $6x^2$ and $7x$ terms that they need by trial and improvement. Sometimes this will lead to a correct expression, but one where the constant term is outside the range. For example:

$$6x^2 + 7x - 20 = (2x + 5)(3x - 4)$$

$$\text{or, } 6x^2 + 7x - 24 = (2x - 3)(3x + 8)$$

If working in this way, students will need to notice that the brackets could be:

$$\text{either } (2x \pm \square)(3x \pm \square)$$

$$\text{or } (6x \pm \square)(x \pm \square)$$

They may also realise that $6x^2 + 7x = x(6x + 7)$ is a possibility, which might be more obvious if they write it as $(x + 0)(6x + 7)$. This is either a very easy or a very difficult example to notice!

Students will need to be systematic in their approach if they want to be sure of obtaining all the possible solutions. They will see that "most" quadratics that they randomly generate are not factorisable, which is a useful thing to realise.

Discussion

You could conclude the lesson with a plenary in which the students talk about the examples that they found and how they went about it. Discussing things that didn't quite work – and why they didn't – may also be useful.

Q. Did you find any that worked? How did you do it? What other ones did you try? Which ones didn't work? Why didn't they work? Did you notice anything about quadratics while you were working these out?

The complete set of solutions is:

$6x^2 + 7x = x(6x + 7)$
$6x^2 + 7x + 2 = (2x + 1)(3x + 2)$
$6x^2 + 7x - 3 = (2x + 3)(3x - 1)$
$6x^2 + 7x - 5 = (2x - 1)(3x + 5)$
$6x^2 + 7x + 1 = (6x + 1)(x + 1)$
$6x^2 + 7x - 10 = (6x - 5)(x + 2)$

but of course this doesn't include things like reversing the order of the pairs of brackets, or writing $-(3x + 5)(1 - 2x)$ instead of $(2x - 1)(3x + 5)$.

Additional resources

See this Resourceaholic article for general approaches to factorising non-monic quadratics, as well as this less-well-known but interesting approach given in Lyszkowski's A simple method for factorizing expressions of the form $ax^2 + bx + c$ (Mathematics in School, 28(1), 34).

Stretch them further

There should be plenty here to stretch all students if they generalise the '6' and the '7' in the given expression to other numbers, but if anyone needs further challenges they could explore the alternative of finding integers to go in the box of this expression so that it factorises:

$$6x^2 + \square x + 7$$

This turns out to be quite a different problem, with only finitely-many solutions, if we restrict ourselves to integers. Since the constant terms in each pair of brackets must be either 1 and 7, or −1 and −7, it turns out that there are just eight possible solutions for the box number: ±13, ±17, ±24 and ±43. (But what happens if the '7' in the expression is changed to some other number?)

Download this lesson plan and task sheet as PDFs here.

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