MAKING PRODUCTS

INVESTIGATING PLACE VALUE BY MOVING AROUND THE DIGITS IN TWO NUMBERS CAN GIVE STUDENTS A STRONGER UNDERSTANDING OF MULTIPLICATION, SAYS COLIN FOSTER

WHY TEACH THIS?

Multiplication is an essential part of any student's understanding of mathematics. Getting a sense of place value enables students to estimate the answer and use technology sensibly.

Multiplication is a fundamental operation in mathematics which every student needs to understand and be comfortable with. Although nowadays we have computers and calculators to do routine calculations for us, we still need to understand what multiplication does and when to use it. It is also important to have a sense for what size an answer should be, so that we can make useful estimates when accuracy is not essential and spot errors when using technology. In this lesson, students have to try to make the maximum product possible by making two numbers from the digits 1 to 9 and multiplying them together. This task allows students to generate lots of practice through trial and error but also focuses their thinking on place value and the optimal positions for the highest-value digits.

STARTER ACTIVITY

Q. I'm thinking of two positive integers which multiply to make 1000. Neither of them contains a zero. What could they be?

Students might need reminding what an integer is (a whole number). They might try for a while and think that it is impossible, but there is a solution: 8 × 125. As soon as someone gets the answer, repeat the question with 1 000 000 instead.

Q. Can anyone find any more of these?

There are some nice patterns here:

$$2 \times 5 = 10$$
$$4 \times 25 = 100$$
$$8 \times 125 = 1000$$
$$16 \times 625 = 10\,000$$
$$32 \times 3125 = 100\,000$$
$$\ldots$$

Each time, we multiply the first factor by 2 and the second factor by 5, so we multiply the whole product by 10. In general $10^n = 2^n \times 5^n$, so $1\,000\,000 = 2^6 \times 5^6 = 64 \times 15\,625$. Another way to get this is to notice that 1 000 000 is $(1000)^2$, so we need 8^2 multiplied by 125^2.

Neither 2^n nor 5^n contains any zeroes for $n < 8$. From $n = 8$ you can continue the same puzzle provided that you say that neither number "**ends in**" a zero.

MAIN ACTIVITY

Q. *Using the digits 1, 2, 3, 4 and 5, make two numbers which multiply to give the biggest possible answer. No calculators, please!*

There are several things to think about, such as whether it is better to have

1-digit × 4-digit
or
2-digit × 3-digit.

Then students need to work out where to place their most valuable digits; i.e., 5 and 4.

There is much opportunity for experimenting, which will lead to students carrying out lots of multiplications. Unlike when doing ordinary exercises, students will be thinking about place value and exploring what happens when the same digits are in different locations. They will also be much more concerned about getting the correct answers. Any large wrong answers are likely to be noticed quickly by other students!

Students may be quite surprised that a product such as 543 × 21 doesn't give anywhere near to the largest possible answer!

If using all the digits from 1 to 5 is too daunting to begin with, students could start with just 1, 2, 3 and 4, or maybe even just 1, 2 and 3. Similarly, if anyone finishes they could try using digits 1 to 6 or even 1 to 9.

DISCUSSION

You could conclude the lesson with a plenary in which the students talk about their answers and how they got them.

Q. *What's the biggest product that you got using the numbers 1 to 5? Don't tell us* ***how*** *you got it – just tell us the answer that you got.*

Write on the board the numbers that you are given.

Q. *Did anyone else get that number? Did anyone get a bigger number?*

Eventually, students can say the numbers that they multiplied to get the answer and students can discuss why the answers come out in the order that they do.

Q. *Why do you think that product came out larger than that one?*
Some findings could be:

1. To get the maximum product, the two numbers need to be of similar size, so 2-digit × 3-digit is going to be better than 1-digit × 4-digit.
2. The digits in each number must decrease from left to right, as the larger digits benefit more from being further to the left.

This means that we must have 5.... × 4.....
Now we have to place the 3 and the 2, and so we have a choice of

53... × 42...
or
52... × 43...

This is the hard bit! The second one is going to give a ***greater*** product, because the gap of 9 between 43 and 52 is smaller than the gap of 11 between 42 and 53. The closer the numbers are together, the greater their product will be. This makes sense if you think about maximising the area of a rectangle with a given perimeter – a square is the best you can do.

So the rule is:

1. Place the digits so that the numbers that you have created (so far) are as close together as possible.

This leads to:

52... × 43...
52 × 431.

By similar reasoning, the maximum products for other sets of digits are:

123	3 × 21 = 63
1234	32 × 41 = 1 312
12345	52 × 431 = 22 412
123456	542 × 631 = 342 002
1234567	742 × 6531 = 4 846 002
12345678	7642 × 8531 = 65 193 902
123456789	9642 × 87531 = 843 973 902

Thinking in the opposite kind of way, the smallest possible products will be:

123	1 × 23 = 23
1234	1 × 234 = 234
12345	1 × 2345 = 2345

etc.

which is actually easier.

INFORMATION CORNER

ABOUT OUR EXPERT

Colin Foster is an assistant professor in mathematics education in the School of Education at the University of Nottingham. He has written many books and articles for mathematics teachers (see www.foster77.co.uk).

ADDITIONAL RESOURCES

FOR MORE INTERESTING FACTS AND PUZZLES CONCERNING THE DIGITS 1 TO 9, GO TO OW.LY/LNBB300ZXPL

STRETCH THEM FURTHER

IF MORE THAN ONE MULTIPLICATION SIGN IS ALLOWED (OR OTHER OPERATIONS THAN MULTIPLICATION ARE PERMITTED), THEN THE PROBLEM BECOMES HARDER. FOR EXAMPLE, 9 × 8 × 7 × 6 × 5 × 4 × 3 × 2 × 1 = 9! ("9 FACTORIAL"), WHICH IS 362 880, IS MUCH SMALLER THAN THE PRODUCT 9642 × 87531 = 843 973 902 GIVEN ABOVE, SO USING EIGHT MULTIPLICATION SIGNS DOES *NOT* GIVE US A LARGER ANSWER. SO MORE MULTIPLICATION SIGNS DON'T NECESSARILY LEAD TO A BIGGER ANSWER.

WHAT IF YOU NEED TO MAKE *THREE* NUMBERS FROM THE DIGITS 1 TO 9 AND USE *TWO* MULTIPLICATION SIGNS? WHAT'S THE LARGEST VALUE YOU CAN MAKE THEN?

ANOTHER CHALLENGE IS TO USE THE 9 DIGITS TO MAKE TWO EQUAL PRODUCTS; FOR EXAMPLE, 23 × 158 = 46 × 79. CAN YOU FIND THE OTHER 10 POSSIBLE SOLUTIONS? THE ANSWERS ARE HERE: OW.LY/NGLY300ZXNP