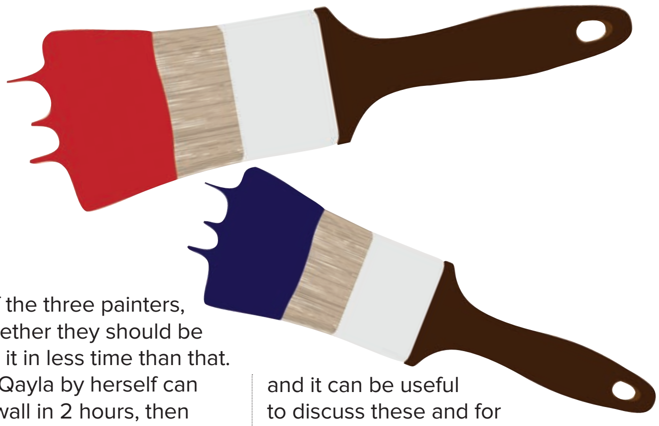
PAINTING A WALL

EVERYDAY LIFE IS FULL OF RATES AT WHICH THINGS HAPPEN, BUT HANDLING THEM CORRECTLY CAN BE TRICKY, SAYS COLIN FOSTER...

Do jobs get done faster when more people help? One proverb says that "Many hands make light work", but another says "Too many cooks spoil the broth"! This lesson is about calculating how long it will take to paint a wall if several people work at it at the same time. Will it take less time with more people helping? And if so then how much less? Students need to make assumptions that are realistic to the real-world situation, but they also have to make assumptions that will make the calculations easy to do and give an answer that is close enough for practical purposes. Students often find calculations involving rates difficult. They may be tempted to add up the *times* rather than the *rates*, and this provides an opportunity for discussion and for deciding whether or not different possible answers obtained are reasonable.

WHY TEACH THIS?

Sometimes the obvious way to work something out doesn't give the right answer. This lesson presents students with a simple-sounding but challenging problem to work on.



STARTER ACTIVITY

Q. We're going to work on a problem about three people painting a wall.

Working on her own, Qayla can paint a wall in 2 hours.
Moses is a bit slower than Qayla and takes 3 hours to paint the same wall.
Leillah is even slower than Moses and takes 6 hours to paint the same wall.
How long will it take if they all paint the wall together at the same time?

A task sheet containing all of the problems in this lesson is available at teachwire.net/painting-a-wall

Give students some time to think about this on their own and/or in pairs. It's likely that some students will add together the three times in the question and get 2 + 3 + 6 = 11 hours. If some do that, you could ask:

Q. Should your answer be more or less than 6 hours? Why?

Students should realise that if Leillah can paint the wall all by herself in 6 hours, and she is the slowest of the three painters, then altogether they should be able to do it in less time than that. Indeed, if Qayla by herself can paint the wall in 2 hours, then the answer must be less than 2 hours. However, there are lots of assumptions needed to answer a question like this. Students might comment on this, or you could ask:

Q. What things do you assume when you answer this question?

It isn't guaranteed that three people will be faster than one – they might get in each other's way, or stand around chatting or arguing or splashing each other with paint! So to make things simple we assume that everyone works at the same speed when together as they would if working on their own. We also assume that they work at the same speed throughout and don't get faster with practice or slower with fatigue, and that no one takes any breaks. There are also assumptions about other factors, such as the wall being uniform, and it can be useful to discuss these and for students to say how much of a difference they think varying the different assumptions might make to the answer.

Since the answer must be less than 2 hours, students might guess at 1 hour. (They also might obtain this by subtracting 6 – 3 – 2 = 1 hour, which gives the right answer but not for the right reason!) They can check this answer by realising that since Qayla can paint the whole wall in 2 hours, she will paint $\frac{1}{2}$ of the wall in one hour. Similarly, Moses will paint $\frac{1}{3}$ of the wall and Leillah will paint $\frac{1}{6}$ of the wall. So altogether, in the same hour, together they will paint $\frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1$ wall! So this means that the answer is 1 hour.

We were lucky, because our guess proved to be right. But how would we do it if the answer weren't as simple as 1 hour?

MAIN ACTIVITIES

Q. We're going to change the problem a bit and see if you can still find a way to solve it.

Let's suppose that Qayla is tired now, and so is working at exactly half her usual speed, but Leillah and Moses are still working as quickly as before.
How long will it take the three of them to paint the wall now?

Let students attempt this problem in pairs and see how they get on. They should see that the answer will be more than 1 hour now, as Qayla is working more slowly than before, but it can't be more than 3 hours, because Moses on his own can paint the wall in 3 hours – and he has help! So students might think that the answer must be 2 hours, since that is between 1 hour and 3 hours. But if they calculate how much wall has been painted after 2 hours they will find that it is more than the whole wall: $\frac{2}{4} + \frac{2}{3} + \frac{2}{6} = 1\frac{1}{2}$, meaning that a wall and a half has been painted after 2 hours.

Students could start again with a new guess now – or they might realise that one wall is $\frac{2}{3}$ of $1\frac{1}{2}$ walls, so they can now calculate $\frac{2}{3}$ of 2 hours, which is 1 hour and 20 minutes. Alternatively, they might work out the situation after 1 hour, which is $\frac{1}{4} + \frac{1}{3} + \frac{1}{6} = \frac{3}{4}$ of a wall painted, and then calculate that therefore $1\frac{1}{3}$ hours will be needed to complete the entire wall – again, 1 hour and 20 minutes.

Students who get an answer could make up their own problem by changing some of the times in the question. For example:

After lunch, Qayla is full of energy and now paints at twice her usual speed (i.e., she can paint the wall by herself in 1 hour).
How long will it take them all to paint the wall now?

(Maybe these three people keep painting the same wall again and again with another coat of paint, or maybe there are several identical walls!) This time it will take them 40 minutes.

Q. Now I want you to look at this data on three different teams. Without doing any calculations, which team do you think would be the fastest at painting the wall?

Team A		*Team B*		*Team C*	
Iman	1 hour	Aga	1 hour	Jasper	2 hours
Faridah	2 hours	Zaina	2 hours	Malika	2 hours
Ayub	6 hours	Sharifah	10 hours	Louise	3 hours

A task sheet containing this data is available at teachwire.net/painting-a-wall

Q. Now I want you to calculate to be sure which is the fastest team.

Students may be surprised at the answer.

STRETCH THEM FURTHER

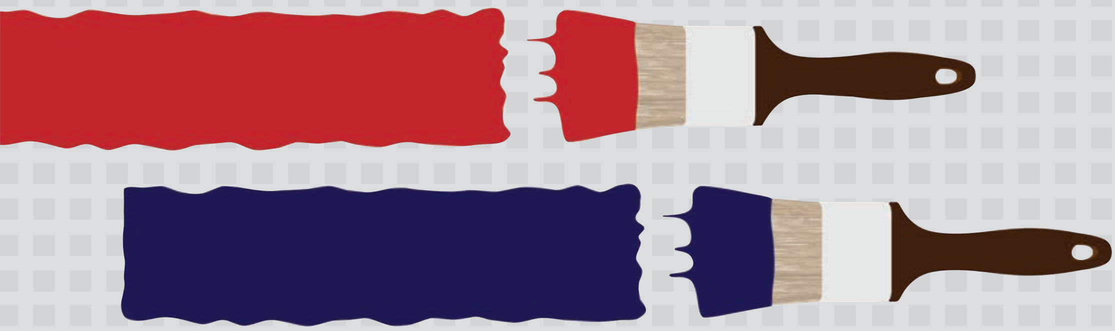
Making up a problem like the ones in this lesson can be a good challenge.

Suppose that there are three people, each working at a different speed, and altogether they take $1\frac{1}{2}$ hours to paint the wall. What could the individual times be?

Another challenging problem is this one:

Charlotte and Carmen can paint a wall together in 45 minutes. Charlotte and Cara can paint the same wall in 54 minutes. Carmen and Cara can paint the wall in 2 hours and 15 minutes. How long will it take if all three of them paint the wall together?

(Before working out the answer, students could try to order the three people according to speed: Charlotte > Carmen > Cara.) The answer is about $41\frac{1}{2}$ minutes, and their individual times are Charlotte: 1 hour; Carmen: 3 hours; Cara: 9 hours.



DISCUSSION

You could conclude the lesson with a plenary in which the students talk about their answers and how they got them.

Q. Which team do you think is the fastest? Can you convince us that they are the fastest team? What calculations did you do? Did anything surprise you? Can you explain why your team is fastest without giving any calculations?

It turns out that Team A is the fastest team (36 min), followed by Team B ($37\frac{1}{2}$ min), followed by Team C (45 min). Students may find this counterintuitive, because Team B looks very disadvantaged, with a 10-hour slowcoach, compared with Team C, whose slowest person is only 3 hours. However, the 1-hour person in Teams A and B does most of the work in those teams, so changing a 6-hour person in Team A for a 10-hour person in Team B doesn't make much difference, as neither of those people will be doing much of the painting. On the other hand, changing a 1-hour person for a 2-hour person makes a dramatic difference to the overall time. Students may have tacitly assumed that the three people in each team each do a third of the work, which is not the case. The amount of painting that they do is inversely proportional to the times that they take to paint a wall.

INFORMATION CORNER

ABOUT OUR EXPERT

Colin Foster is an Assistant Professor in mathematics education in the School of Education at the University of Nottingham. He has written many books and articles for mathematics teachers (see www.foster77.co.uk).