WHY TEACH THIS?

Our modern world is full of geometrical images of various kinds. For example, the designs you see on wallpaper may involve transformations such as reflections, rotations and translations. Making sense of how geometrical transformations combine can help students to develop their problem-solving skills and deepen their spatial understanding.

REPEATED ROTATIONS

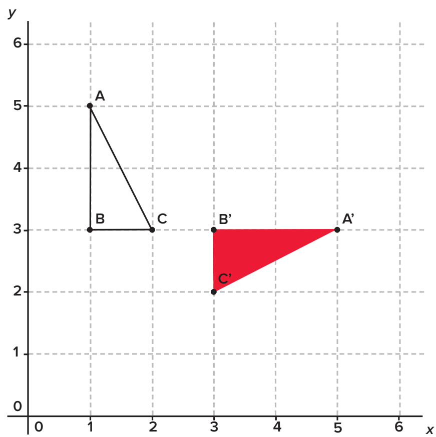
EXPLORING TWO SUCCESSIVE GEOMETRICAL TRANSFORMATIONS CAN GIVE STUDENTS INSIGHT INTO HOW THEY COMBINE, SAYS COLIN FOSTER...

The topic of geometrical transformations is often treated as a collection of skills to be taught, practised and performed without deep understanding. Students simply memorise a process for carrying out reflections, rotations, translations and enlargements, but then only rarely go on to use this to help them pose or solve any mathematical problems. In this lesson, students explore what happens when you rotate a shape and then rotate it again. If the same centre of rotation is used for both rotations, then the rotations simply add up, but if the centres of rotation are different then something much more interesting happens. Students can explore varying the original shape and the positions of the centres of rotation, and perhaps also the angle and sense (clockwise or anticlockwise) of rotation – although initially it is helpful to fix both rotations as 90° clockwise. Is the overall result always a rotation? (Not necessarily, in general, but with two 90° rotations it is!) How does the centre of rotation for the *combined* rotation relate to the centres of rotation for the two separate rotations? There is lots for students to explore, whether using pencil and paper or software such as the free program *Geogebra*.

STARTER ACTIVITY

Two triangles

Q. Look at the two triangles in this picture. What is the same and what is different about them?



A task sheet containing these is available at teachsecondary.com/downloads/maths-resources

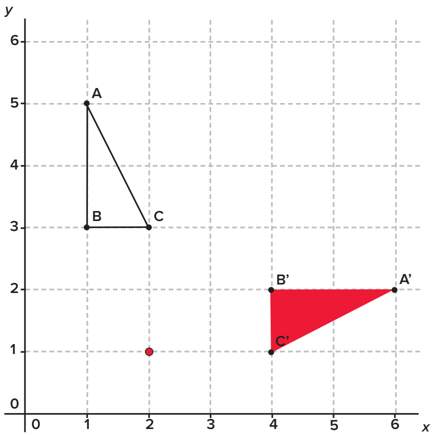
There are some easy things that students can say, such as that the two triangles are the same shape and size and they are different colours and have different orientations. Encourage students to use precise mathematical language (e.g., the word ***congruent***). If they say that the red one is "rotated", encourage them to give as much information as possible: the red triangle *A'B'C'* (the ***image***) is the result of rotating the white triangle *ABC* (the ***object***) by 90° clockwise about the point (2, 2).

It might be quite hard for students to locate the centre of rotation. If they say the wrong point, ask or help them to draw where the triangle *ABC* ***would*** go under a 90° clockwise rotation about their point. This may help them to see where the actual centre of rotation is.

Q. What would happen to the red triangle if I moved the centre of rotation to (2, 1)?

See if students can visualise this without drawing. If not, they could work in pairs to make some rough sketches, or accurate drawings, to help them work it out. Moving the centre of rotation down 1 unit moves the image 1 unit to the right and 1 unit down, as shown in the drawing below.

If you have access to software such as *Geogebra* (available free from **www.geogebra.org/cms/en/**), then students could explore for themselves what happens when you move the centre of rotation to other positions. (They could create their own file or use the one available at **www.foster77.co.uk/One%20rotation.ggb**.)



+ KEY RESOURCE

DON'T MISS THIS...

The MA 2015 Annual Conference Fluency and Understanding - A Mathematically Balanced World at Keele University on 8-10 April 2015.

"A 3-day event that's brilliant and refreshing."

"It will renew your mathematical enthusiasm."

"All your CPD needs covered."

Everyone is welcome at the MA's Annual Conference, endorsed by the NCETM CPD Standard. Nowhere else do you get such a range of workshops and such a rich and varied range of enthusiastic mathematics educators: primary and secondary teachers, PGCE students and university academics, to mention just a few. There's an immense variety of workshops and everyone leaves feeling refreshed, renewed and challenged.

Keynote speakers: Mike Askew, Ruth Merttens, Lynne McClure, Alex Bellos (author of Alex through the Looking-Glass) and Andrew Jeffrey (The Mathemagician).

Find out more at m-a.org.uk/jsp/index.jsp?lnk=120

MAIN ACTIVITY

*Q. Now we are going to rotate a shape 90° clockwise and then rotate the image **another** 90° clockwise. What do you think is going to happen?*

Students could discuss this in pairs and maybe try a couple of examples. They might assume that the centre of rotation is going to be the same for the two rotations, in which case the overall result will be a 180° rotation about this point. (For a 180° rotation we don't need to say "clockwise" or "anticlockwise"!) The question is more interesting if the centres of rotation for the two rotations are different.

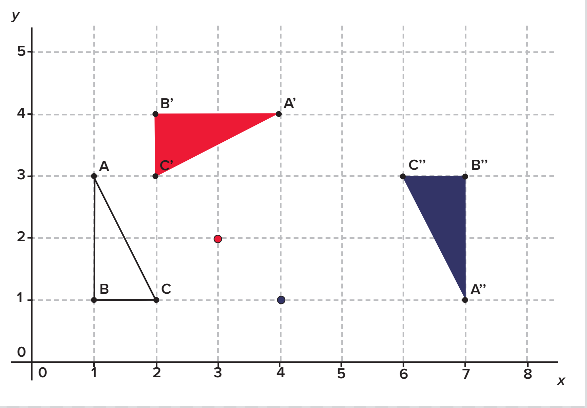
Q. What if the centres of rotation are different for the two rotations. Can you find out what happens?

This is hard and will take quite a lot of exploration. In the process, students will gain a great deal of practice at performing the rotations and will need to be careful, systematic and thoughtful about what they are doing. They might generate some data and look for patterns or think more deeply about what is happening and why.

If students are struggling to understand the task, you could do one together as a whole class. For example, in the diagram below the white triangle is rotated 90° clockwise about the red point to give the red triangle, and then the red triangle is rotated 90° clockwise about the blue point to give the blue triangle. The question is how to do we get, in one step, from the white triangle to the blue triangle?

The answer is that *ABC* is rotated 180° about the point (4, 2). We could ask whether this overall transformation would be the same if we moved *ABC* to a different position? Or if we changed its shape? And how does the point (4, 2) depend on the positions of the two centres of rotation (the red and blue dots)? There is lots here for students to explore, whether on paper or by using *Geogebra*. (They could create their own file or use the one available at **www.foster77.co.uk/Two%20rotations.ggb**.)

Students who need further challenges could investigate what happens if you go anticlockwise instead of clockwise, or if you make one of the rotations 180° instead of 90°.



ADDITIONAL RESOURCES

A VERY NATURAL ENVIRONMENT FOR EXPLORING THESE PROBLEMS IS THAT PROVIDED BY *GEOGEBRA*, WHICH IS AVAILABLE FREE OF CHARGE FROM **WWW.GEOGEBRA.ORG/CMS/EN/**. SUITABLE FILES FOR THIS LESSON ARE AVAILABLE AT: **WWW.FOSTER77.CO.UK/ONE%20 ROTATION.GGB** AND **WWW.FOSTER77.CO.UK/TWO%20 ROTATIONS.GGB**. A TASK SHEET CONTAINING THE DIAGRAMS FOR THIS LESSON IS AVAILABLE AT **TEACHSECONDARY.COM/DOWNLOAD/ MATHS-RESOURCES**

INFORMATION CORNER

ABOUT OUR EXPERT

Colin Foster is an Assistant Professor in mathematics education in the School of Education at the University of Nottingham. He has written many books and articles for mathematics teachers (see www.foster77.co.uk).

DISCUSSION

You could conclude the lesson with a plenary in which students talk about what they have found out and learned. They might have generalised from what they have seen in the examples that they have tried so that they have conjectures about how the centre of rotation for the overall 180° rotation relates to the positions of the red and blue dots. Or they might have reasons why they think their conclusions are correct.

If you take a general point (x, y) and rotate it 90° clockwise about the point (a, b), its image will be $(y - b + a, a - x + b)$. It follows that if you then rotate *this* point another 90° clockwise about the point (c, d) it will end up at $(a + b + c - d - x, b + c + d - a - y)$. These two rotations are together equivalent to a single 180° rotation about the point $(\frac{1}{2}(a + b + c - d), \frac{1}{2}(b + c + d - a))$. For example, if the two centres of rotation are (3, 2) and (4, 1) then the centre of rotation for the overall 180° rotation will be (4, 2).

If students have access to *Geogebra*, they will be able to see that when you keep the centres of rotation fixed but move the original triangle, the other two triangles move, but the overall transformation from the original triangle to the final triangle stays the same. If they change the original triangle to some other shape, again the overall transformation remains the same.

STRETCH THEM FURTHER

CAN STUDENTS WORK OUT A GENERAL METHOD FOR FINDING THE CENTRE OF ROTATION WHEN GIVEN THE OBJECT AND THE IMAGE? WHAT IF THE CENTRE OF ROTATION IS NOT AT A POINT WITH INTEGER COORDINATES OR THE SHAPES ARE NOT DRAWN ON A COORDINATE GRID?

STUDENTS WILL NEED TO CONSTRUCT THE PERPENDICULAR BISECTORS OF CORRESPONDING POINTS; I.E., FOR THE TRIANGLES *ABC* AND *A'B'C'* THEY WOULD NEED TO DRAW THE PERPENDICULAR BISECTORS OF *AA', BB'* AND *CC'*. WHERE THESE INTERSECT MUST BE THE CENTRE OF ROTATION. CAN STUDENTS JUSTIFY WHY THE THREE LINES MUST BE CONCURRENT (I.E., INTERSECT AT A SINGLE POINT)? CAN THEY EXPLAIN WHY THIS POINT MUST BE THE CENTRE OF ROTATION?

ANOTHER TASK FOR CONFIDENT STUDENTS IS TO ASK WHETHER TWO SUCCESSIVE ROTATIONS WILL ALWAYS LEAD TO A ROTATION? ACTUALLY, THEY WON'T NECESSARILY. FOR EXAMPLE, ROTATING A SHAPE 180° ABOUT (3, 2) AND THEN ROTATING THE IMAGE 180° ABOUT (4, 1) IS EQUIVALENT TO A ***TRANSLATION*** $\binom{2}{-2}$, NOT A ROTATION. WHEN DOES THIS HAPPEN AND WHY?