WHY TEACH THIS?

The world around us is full of uncertainties. How can we know how likely something is to happen? The answer is that we need data.

ROLLS OF THE DICE

ESTIMATING HOW LIKELY SOMETHING IS TO HAPPEN CAN BE HARD. CARRYING OUT A PROBABILITY EXPERIMENT CAN BE INFORMATIVE IF WE KNOW HOW TO INTERPRET THE DATA, SAYS COLIN FOSTER

People's intuitive estimates of how likely something is to happen are often very far away from the truth. People frequently vastly over-estimate or under-estimate many everyday risks, and this can lead to poor decision making. One way to estimate the probability of something happening is to carry out an experiment and obtain some data on how many times it occurs out of a certain number of trials. The proportion of occurrences (relative frequency) gives an estimate of the probability, and this estimate generally improves the more trials that you do. In this lesson, students will carry out an experiment to estimate the probability that when three ordinary dice are thrown, they will show consecutive numbers. Students will explore how their estimates change as more and more throws are included, leading to an appreciation of the law of large numbers.

STARTER ACTIVITY

*Q. What does it mean if a die is **biased**?*[1]

Students may not be familiar with the word "biased", which means unfair. A biased die is more likely to land on some faces than on others. For example, the die might not be a perfect cube, and some faces or edges might be worn down different amounts. There might be a piece of metal deliberately buried inside it at some position away from the centre, so as to weight it in one direction. More simply, there might be two faces marked with a 5, say.

All dice are at least a little bit biased, since a truly unbiased die is an ideal never perfectly attained in real life. In fact, even if the die were perfect, it still might be thrown in such a way as to bias the outcome (some people claim to be able to make "controlled throws").

*Q. How would you decide whether a die is biased or not? Can you tell a biased die if you throw it just once? What if it's **very** biased? What if you throw it twice? How many times would you need to throw it?*

You might suspect that a die is biased just by looking at it, or by rolling it in the palm of your hand, but unless it's really obvious you would need to throw it. No matter how great the bias, throwing a die just once tells you very little, because whichever number you obtain could have a large or a small probability. Students will realise that they could need to throw the die many times in order to detect a bias. The **law of large numbers** says that the average of the outcomes from a number of throws of the die will settle down to a particular value as you throw it more times. You can get as close to this value as you like if you make enough throws. How many throws you need depends on how precise you want to be about the value.

[1] Sometimes people use 'die' as the singular of dice, but dictionaries now seem to say that 'dice' is OK

MAIN ACTIVITY

Q. *Imagine you throw three dice at once. How likely do you think you are to get three **consecutive** numbers, like 3, 4 and 5?*

Students might need to clarify the meaning of "consecutive". They are unlikely to be able to calculate the answer, but they could estimate roughly how big they think it might be. They will probably realise that it is going to be less than 50%, for instance. It doesn't matter what guesstimates you get, as students will be able to test this out in the next part of the lesson and see how accurate they were.

Q. *We're going to try it lots of times and see what happens. How should we record our results?*

One possibility is a tally chart like this:

Not all consecutive	*All consecutive*
卌 卌 III	卌 II

But it's interesting also to keep track of the ***order*** of what happens by using a chart like this instead:

Throw	*Not all consecutive*	*All consecutive*
1	✓	
2		✓
3	✓	

etc.

Students should work in pairs, one person throwing the three dice (all together) and the other recording whether three consecutive numbers are obtained or not. Half way through, they could swap over.

Q. *You have five minutes to do as many throws of the dice as you can. The more results we get, the better!*

After five minutes, stop the class. (It might be easier to collect up the dice at this point, so that they don't cause a distraction, as they won't be needed for the next part of the lesson.)

Q. *What do you think your results are telling you? Why?*

Students could comment on the patterns that they see in their results and how likely they think it is to get all consecutive dice.

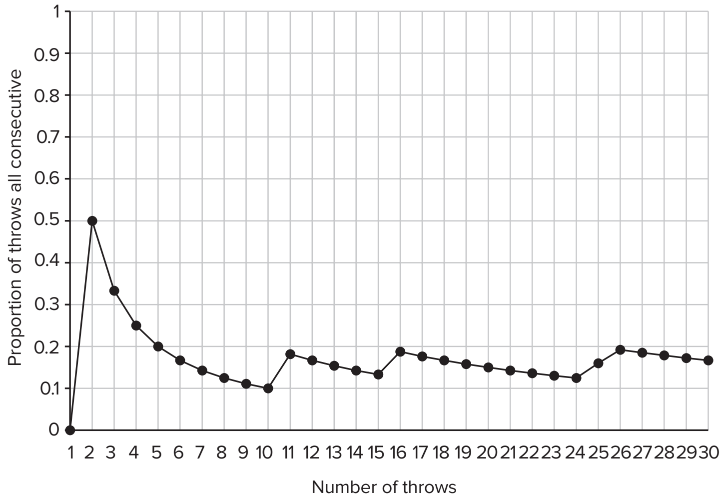
How does this confirm or challenge what they predicted at the start of the lesson?

Q. *Now add an extra column to your results table where you work out after each throw the **proportion** of all-consecutives so far:*

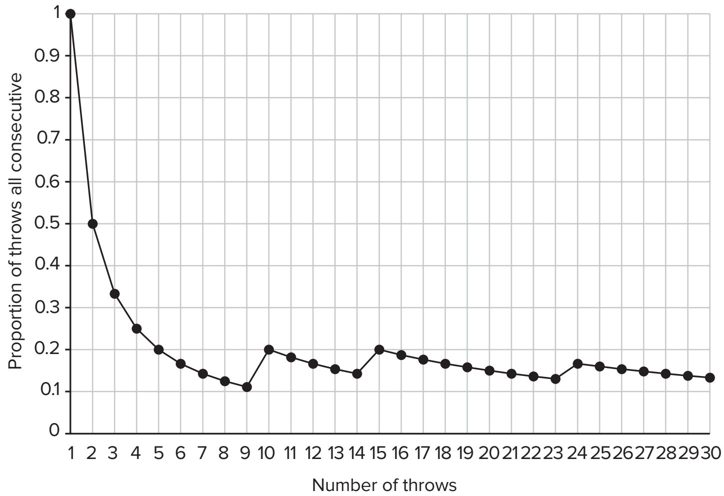
Throw	*Not all consecutive*	*All consecutive*	***Proportion all consecutive so far (3 dp)***
1	✓		**0.000**
2		✓	**0.500**
3	✓		**0.333**

Q. *Now draw a graph of the last column against the first.*

It could look something like this:



Or maybe like this, if the first throw gives consecutive dice:



Q. *What do you think will happen if we keep throwing the dice more and more times?*

Although these two graphs look completely different at the beginning, both settle down after a while to roughly the same value, somewhere between 0.1 and 0.2.

DISCUSSION

Q. *Each pair has collected some data, but if we put all our data together we ought to be able to get a more accurate answer.*

Enter each group's totals into a spreadsheet to calculate the total number of consecutive throws along with the proportion of consecutive throws out of all the throws made so far. A suitable spreadsheet is available at teachwire.net (see ow.ly/B06I303uJCz), which will draw a graph as you go of the proportion of consecutive throws against the total number of throws.

Throw	*Not all consecutive*	*All consecutive*	*Total number of throws so far*	***Proportion of throws all consecutive (3 dp)***
1	44	5	49	**0.102**
2	48	7	104	**0.115**
3				

etc.

Try to get students to predict how each group's result will affect the graph.

Q. *What's going to happen when I put in the next result? Will it go up or down?*

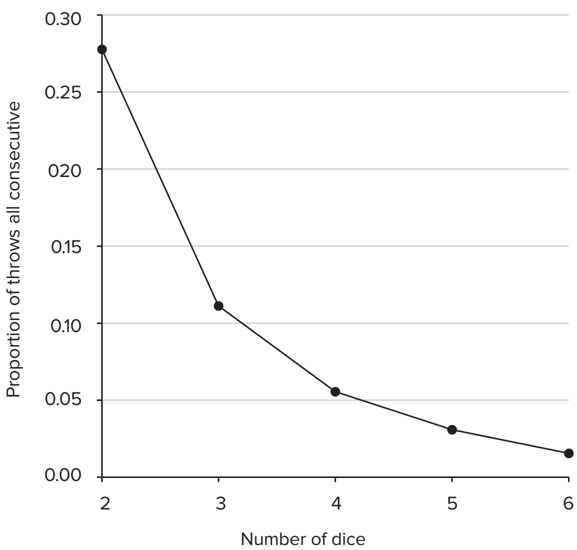
Q. *Describe to the person next to you what is happening to the graph? What is it telling us?*

Students will see that the more throws we have the more stable the graph becomes, and the closer it seems to get to a particular value. If you have enough throws in total, you should end up quite close to the theoretical prediction of $1/9$ (see 'Stretch them further' box for an explanation).

Q. *What do you think would happen with more or fewer dice, if* ***all*** *the dice have to be consecutive for it to count?*

Students should realise that it will be much easier for two dice to come out consecutive than for, say, four dice. If there is time, students could repeat their experiments with different numbers of dice (different pairs could try 2, 4, 5 and 6 dice, if you have enough dice). In general, with n dice (where n is from 2 to 6) the probability that all the numbers will be consecutive is $\frac{n!(7-n)}{6^n}$. This gives the values in the table below, which decrease with each additional die, as you can see in the graph.

Number of dice	*Probability of throws all consecutive (correct to 3 decimal places)*
2	$5/18$ = 0.278
3	$1/9$ = 0.111
4	$1/18$ = 0.056
5	$5/162$ = 0.031
6	$5/324$ = 0.015



INFORMATION CORNER

ABOUT OUR EXPERT

Colin Foster is an assistant professor in mathematics education in the School of Education at the University of Nottingham. He has written many books and articles for mathematics teachers (see www.foster77.co.uk).

ADDITIONAL RESOURCES

A SPREADSHEET FOR ENTERING THE RESULTS IS AVAILABLE AT TEACHWIRE. NET. PUPILS INTERESTED IN DICE THROWING MIGHT LIKE TO READ WWW.INSIDESCIENCE.ORG/ BLOG/2012/09/12/DICE-ROLLS-ARE-NOT-COMPLETELY-RANDOM.

STRETCH THEM FURTHER

CONFIDENT STUDENTS COULD THINK ABOUT HOW THEY MIGHT CALCULATE THE PROBABILITY EXACTLY, ALTHOUGH THIS IS HARD. WITH THREE DICE, THERE ARE FOUR POSSIBLE STRINGS OF THREE CONSECUTIVE NUMBERS (1-2-3, 2-3-4, 3-4-5 AND 4-5-6). (ONE WAY TO SEE THIS IS TO NOTE THAT THERE ARE FOUR DIFFERENT NUMBERS THAT CAN BE IN THE MIDDLE.) EACH OF THESE COULD APPEAR IN SIX DIFFERENT ORDERS (FOR EXAMPLE, 1-2-3 COULD ALSO BE 1-3-2, 2-1-3, 2-3-1, 3-1-2 OR 3-2-1), MAKING 4 × 6 = 24 WAYS IN WHICH YOU COULD GET CONSECUTIVE NUMBERS. THE TOTAL SAMPLE SPACE HAS 6^3 = 216 MEMBERS, SO THE PROBABILITY OF GETTING CONSECUTIVE DICE WILL BE $24/216$ = $1/9$. (THIS IS EASIER TO THINK THROUGH FOR TWO DICE, AND HARDER FOR MORE THAN THREE DICE.)