The probability distribution for a biased spinner

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Abstract   This article advocates biased spinners as an engaging context for statistics students. Calculating the probability of a biased spinner landing on a particular side makes valuable connections between probability and other areas of mathematics.

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Students sometimes feel that probability and statistics are rather disconnected from the rest of the mathematics curriculum, having more to do with other subject areas than they do with mathematics itself. In my view, links *within* mathematics are just as important as good cross-curricular contexts, and relating probability to other areas of the subject is a good way of bringing some unity to students' mathematical experience.

Spinners (figure 1) are ubiquitous in probability teaching. *Unbiased* spinners consisting of regular n-sided polygons with a spike through the centre of mass are frequently used in children's games to give n equally likely outcomes, and so tend to be familiar objects. Of course, in practice, no spinner in the real world will be perfectly ideal, as the spike will not pass precisely through the centre of mass or make exactly a right angle, and the polygon will not be perfectly regular and uniform, not to mention various difficulties associated with the spinning. In this sense, all spinners are *biased*. However, it is possible to design and make *deliberately* biased spinners, and the calculations necessary can be interesting and draw on mathematics that students are likely to know about from elsewhere in their mathematics education.

A biased spinner can be made from a regular polygon by ensuring that the spike does not pass through the centre of mass. However, such a spinner will wobble as it spins and its behaviour may be hard to model mechanically. An alternative is to consider an *irregular* polygon with the spike passing through its centre of mass. These can be easily constructed from thick cardboard, with a needle pushed through as the spike,

allowing students to test their calculations by spinning the spinner a certain number of times. They can then compare the relative frequencies of the spinner landing on the different sides with the theoretical probabilities that they have calculated. Accounting for whatever differences are found is an important aspect of mathematical modelling; the original model may need to be refined or even discarded.

Students may offer different ideas about what exactly determines the probability of the spinner landing on a particular side. One possibility is that it is the ratio of the length of the side to the total perimeter of the polygon. This clearly cannot be the full story, since the position of the spike, which is clearly important, does not feature. However, we will see that it can give a very good approximation to the truth when the spike passes through the centre of mass. Another possibility is that the probability is determined by the area of the triangle made by a particular side and the line segments joining its ends to the position of the spike, the probability being the ratio of this area to the area of the whole polygon. This might be suggested by parallels that students may draw with Kepler's *equal areas in equal times* second law of planetary motion, which they may have encountered in their mechanics studies.

A third possibility, which we will pursue here, is that the probability that the spinner will land on a particular side is proportional to the *angle* subtended by that side at the position of the spike. If we place the spike through the centre of mass, then for a triangle this will be the centroid, which is located at the intersection of the medians, which join each vertex to the midpoint
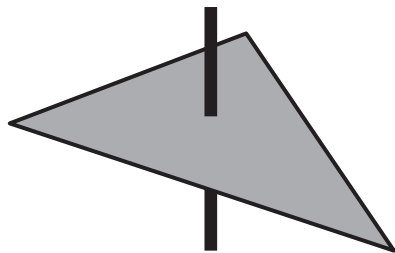
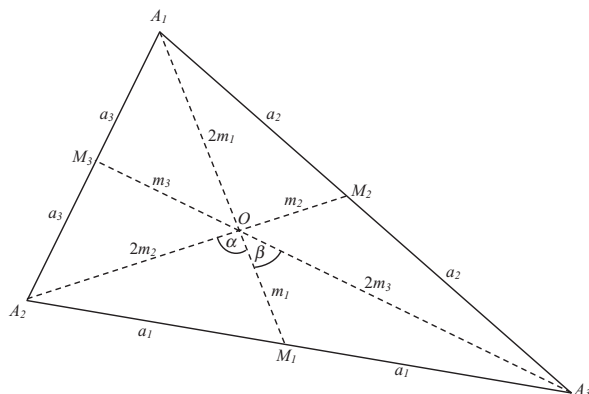Fig. 1. Spinner consisting of a spike through a triangular lamina

Fig. 2. Triangular lamina with medians displayed

of the opposite side. It is a well-known result in pure mathematics that the centroid is positioned two thirds of the way along each median from its vertex.

Students might be best advised to do their initial calculations with a specific triangle (e.g. an isosceles one), but for the teacher's benefit, an analysis of the general case for a triangle is given here.

Consider a spinner consisting of a uniform lamina with a perpendicular spike passing through its centre of mass. We will consider the case where the lamina is a general triangle and establish a relationship between the side lengths and the probability that the (biased) spinner will land on any particular side. Figure 2 shows a general triangle $A_1A_2A_3$ with sides of length $2a_1$, $2a_2$ and $2a_3$ and medians A_1M_1, A_2M_2 and A_3M_3 of length $3m_1$, $3m_2$ and $3m_3$, intersecting at the centroid O. Angles α and β are as shown.

Now we are supposing that the probability that the spinner will land on side A_2A_3, for instance, will be $(\alpha + \beta)/(2\pi)$. We now need to express this in terms of a_1, a_2 and a_3. Using the cosine rule in triangle $A_1A_2M_1$ gives

$$(3m_1)^2 = (2a_3)^2 + a_1^2 - 2(2a_3)a_1 \cos A_2$$

so

$$9m_1^2 = a_1^2 + 4a_3^2 - 4a_1a_3 \cos A_2 \qquad (1)$$

And using the cosine rule in triangle $A_1A_2A_3$ gives

$$(2a_2)^2 = (2a_3)^2 + (2a_1)^2 - 2(2a_3)(2a_1) \cos A_2$$

so

$$2a_2^2 = 2a_1^2 + 2a_3^2 - 4a_1a_3 \cos A_2 \qquad (2)$$

Subtracting equation (2) from equation (1), we obtain

$$9m_1^2 - 2a_2^2 = 2a_3^2 - a_1^2$$

so

$$m_1^2 = \frac{2a_2^2 + 2a_3^2 - a_1^2}{9}$$

In the same way,

$$m_2^2 = \frac{2a_1^2 - a_2^2 + 2a_3^2}{9}$$

and

$$m_3^2 = \frac{2a_1^2 + 2a_2^2 - a_3^2}{9}$$

Now using the cosine rule in triangle OA_2A_3:

$$\cos(\alpha + \beta) = \frac{(2m_2)^2 + (2m_3)^2 - (2a_1)^2}{2(2m_2)(2m_3)}$$
$$= \frac{m_2^2 + m_3^2 - a_1^2}{2m_2m_3}$$

and substituting, we obtain

$$\cos(\alpha + \beta) = \frac{\left(\dfrac{2a_1^2 - a_2^2 + 2a_3^2}{9} + \dfrac{2a_1^2 + 2a_2^2 - a_3^2}{9} - a_1^2\right)}{2\left(\dfrac{\sqrt{(2a_1^2 - a_2^2 + 2a_3^2)(2a_1^2 + 2a_2^2 - a_3^2)}}{9}\right)}$$
$$= \frac{a_2^2 + a_3^2 - 5a_1^2}{2\sqrt{(2a_1^2 - a_2^2 + 2a_3^2)(2a_1^2 + 2a_2^2 - a_3^2)}}$$

Some calculated probabilities for a few simple triangles are given in table 1 (the *angle* model).

Table 1. Theoretical probabilities according to two models (correct to 3 significant figures, where necessary)

Side lengths			Angle model			Length model		
$2a_1$	$2a_2$	$2a_3$	$p(2a_1)$	$p(2a_2)$	$p(2a_3)$	$p(2a_1)$	$p(2a_2)$	$p(2a_3)$
1	1	1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
1	2	2	0.210	0.395	0.395	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{2}{5}$
1	1	$\sqrt{2}$	0.301	0.301	0.398	0.293	0.293	0.414
3	4	5	0.259	0.340	0.401	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{5}{12}$

For comparison, the much simpler process of calculating the ratio of the length of each side to the total perimeter $2(a_1 + a_2 + a_3)$ is also shown in table 1 (the *length* model). These values closely agree and might provide students with an opportunity to analyse the differences.

There is much here for students to explore. Many are surprised that theoretical probabilities can be calculated at all other than in simple symmetrical situations where a small finite number of outcomes are all equally likely. Others realize afresh that *three outcomes* does not necessarily mean *three equally likely outcomes*. This task brings together different aspects of mathematics, allowing students to make rich connections within what they know.

Historical vignette

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Concerning a famine in Lydia, Herodotus remarks:

> [The Lydians] claim also the invention of all the games which are common to them with the Greeks. These they declare that they invented about the time when they colonised Tyrrhenia, an event of which they give the following account. In the days of Atys, the son of Manes, there was great scarcity through the whole land of Lydia. For some time the Lydians bore the affliction patiently, but finding that it did not pass away, they set to work to devise remedies for the evil. Various expedients were discovered by various persons; dice, and huckle-bones, and ball, and all such games were invented, except tables, the invention of which they do not claim as theirs. The plan adopted against the famine was to engage in games one day so entirely as not to feel any craving for food, and the next day to eat and abstain from games. In this way they passed eighteen years.

Reference

Herodotus (c. 440 B.C.E.). *The History of Herodotus*, Book I, Chapter 94, Sections 2–4. Translated by George Rawlinson. http://classics.mit.edu//Herodotus/history.html (accessed 23 June 2011).