What's so hard about algebra?

by **Jeremy Hodgen** and **Colin Foster**

As part of a project funded by the Nuffield Foundation, *Investigating Mathematical Attainment and Progress*, we recently interviewed over 100 low-attaining secondary students about their learning of mathematics. They talked about what they are successful with in mathematics, and also about what they find hard. It will come as no surprise to readers that many of them highlighted 'algebra' as the hardest thing in mathematics. But why is algebra seen as being so hard? Here are a few reasons that we have heard people say, which we have found ourselves beginning to question a little.

1. *It's the letters – students find them very confusing.*

But words contain letters, and they can mostly read. So it can't be letters *per se*.

2. *It's the symbols – they are so abstract. Students find using abstract symbols very hard.*

But '4' is a pretty abstract symbol. It isn't 4 sausages – it's just '4'. And '4 × 5' is three pretty abstract symbols combined, so why is it harder when it is '*ab*'? Weak knowledge of number facts may interfere with students' calculation of something like '4 × 5', but doesn't affect working with '*ab*'. So shouldn't '*ab*' be *easier*?

3. *It's the notation – the fact that we write, for example, 'ab' rather than 'a × b'. These arbitrary notational conventions are non-intuitive.*

True, but the same goes for numbers, surely. Why should '15' mean '10 + 5'? It's a non-obvious convention, but most students get used to it.

4. *It's the deep structure that is difficult – that's really what students find hard.*

But the 'structure' of school algebra is exactly the same as the structure of our number system – isn't that the whole point of algebra? The match between how numbers behave and how letters behave is what makes algebra useful. Things like distributivity and associativity are aspects of the structure of the real numbers. So why is this a reason for *algebra* being hard rather than a reason for *numbers* being hard?

5. *It's the fact that students are taught it procedurally rather than with an emphasis on understanding.*

But the same goes for number work, surely?

We are not saying that these responses are all wrong. But we think it may be a bit more complicated than that. We have begun to suspect that if students really understand number, then symbolic algebra shouldn't be a massive jump. Certainly, there are steps up to the notions of generalized number, variable and covariation, and the Cartesian graph too. But these are steps, rather than great leaps forward. So, perhaps it is what we *do* with algebra that makes it hard. Of course, in school, we often introduce algebra to do things that we can easily do without algebra, such as, "If $a + 2 = 6$, what is a?", and then students might find it hard to see the point of algebra.

We recently carried out an extensive review of research for the *Education Endowment Foundation* to develop a guidance report, *Improving Mathematics in Key Stages Two and Three*, which made eight recommendations (EEF, 2017). Two sub-recommendations relate to algebra: one about understanding procedures and one about recognizing structure. One of the problems with algebra is that to many students the 'rules' seem arbitrary and so are difficult to remember. We want students to be in a position where they can *reconstruct* the procedures and rules for themselves.

It is helpful to think about school algebra as generalized number. The letters/symbols represent numbers or sets of numbers, not objects like apples and bananas. So it is crucial for students to develop a strong foundation in number – including a good understanding of the 'rules' of arithmetic, including:

Commutativity: $a + b = b + a$ and $ab = ba$.

Associativity: $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$.

Distributivity: $a(b + c) = ab + ac$.

These are all about the order in which operations take place, but they are far more than 'BIDMAS', and these 'rules' don't hold for the inverse operations of subtraction and division. We've used algebra to describe these, but it is really important to understand these as rules *about number*. It is also essential to develop students' understanding of equivalence – and the equals sign as meaning "is the same value as". And it's important to know the zero product property (i.e. if the product of two numbers is 0, then at least one of them must be 0).

However, as we've already said, some aspects of algebra do involve a step up in thinking. We often focus on the use of symbols as being the hard part, but actually it is not the symbols themselves that are hard. The step up is in terms of *generalization*. The real power of algebra comes when we use symbols to stand for *sets of numbers*, rather than a specific unknown, and then we use symbols to describe *relationships*. Although this is certainly a step up, we can make this easier by helping students to see the symbols as standing for numbers.

So what might this mean in practice? At KS3, perhaps we might place less emphasis on manipulation and equations and more on expressions and developing meaning. This doesn't mean not doing any manipulation, but we need to encourage students to keep in mind that these letters stand for numbers, and that the rules are not arbitrary. We think *Grid Algebra* (Note 1) is a great resource, particularly at Year 7, because it allows students to explore creating and undoing quite complicated numeric and algebraic expressions. You might also use things like the ICCAMS 'comparing expressions' lesson sequence, which we developed with Dietmar Küchemann and Margaret Brown (Note 2). These lessons focus on how the value of an algebraic expression can vary, and examine the power of the Cartesian graph to represent this. You will also find a huge range of related lessons and activities in the Standards Box by Malcolm Swan (2005).

Textbooks often use perimeter and area representations to help students understand algebraic expressions and relationships. These are really powerful models, provided that students have a reasonably strong sense of area, a concept that (surprisingly, perhaps) many KS3 students struggle with; see, for example, Peter Bryant's excellent review of how students learn geometry and spatial reasoning (Bryant, 2009). Students may well know how to *calculate* the area of a rectangle, but may be less certain what area *is*, or why the 'base times height' formula works. It is worth spending time on this and really getting to grips with the idea that a 12 by 15 rectangle is 12 rows of 15 squares, or 15 rows of 12 squares. It is not so difficult, but we rarely spend long enough on it.

We suspect that many students say that algebra is hard because they have heard other people say that it is hard. A great deal of algebraic thinking is simply generalizing students' understandings of number and, provided they do have a strong understanding of the structure of our number system, the step up to algebra should not be a terrifying leap. But we do need to convince students of the relationship between algebra and number. So, the next time a student makes an error like $(a + b)^2 = a^2 + b^2$, why not try asking them to see what happens when they substitute different numbers for a and b, by 'tracking' what happens to the numbers (Mason, 2018).

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Notes

1. See **https://www.atm.org.uk/shop/Primary-Education---View-All/Grid-Algebra---Site-Licence**
2. This is available at **http://iccams-maths.org/algebra/** The ICCAMS project, *Increasing Competence and Confidence in Algebra and Multiplicative Structures*, was originally funded by the Economic and Social Research Council (ESRC), and is currently funded by the Education Endowment Foundation.

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