The fundamental problem with teaching problem solving

Colin Foster reflects on dilemmas in teaching students how to solve mathematical problems.

I take problem solving to be "engaging in a task for which the solution method is not known in advance" (NCTM, 2000, p. 52). This is the opposite of doing what you have just been taught how to do, and is more than just applying the method you have been shown to a contextual situation. As many people have said, problem solving is what you do when you do not know what to do. I agree with the mathematician Paul Halmos that solving problems is the principal, pre-eminent activity of the mathematician. When considering what constitutes "the heart of mathematics", he concluded that this was not theorems and proofs but problem solving: "the mathematician's main reason for existence is to solve problems [...] therefore, what mathematics *really* consists of is problems and solutions" (Halmos, 1980, p. 519, original emphasis).

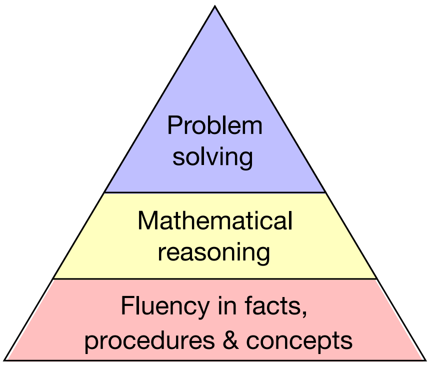

Figure 1: How problem solving depends on reasoning and fluency.

Relating this to the aims of the English national curriculum, I think of fluency and mathematical reasoning, not as ends in themselves, but as means to support students in the most important goal of all: solving problems (see figure 1). Unless students have the necessary fluency in important facts, procedures and concepts, and rich experience in reasoning mathematically, they will struggle to make progress when solving unfamiliar problems. I think this means that although problem solving is central to mathematics, not all mathematics lessons necessarily need to be problem-solving-focused. Students need to spend time on developing their fluency and reasoning if they are to be in a strong position to tackle problems, although there are ways of doing this that can incorporate aspects of problem solving (Foster, 2014).

But, what about those problem-solving lessons? How do we teach students to solve mathematical problems? If problem solving means not knowing in advance how to solve the problem, it follows that the teacher can kill the problem-solving aspect if they teach, or re-teach, the method immediately before giving students the problem. This may seem like a helpful thing to do in the short term, and it is likely to help the students to solve that particular problem, but it is unlikely to support their long-term ability to solve unfamiliar problems, where hints are not available. I have seen lessons where the teacher has prepared a problem-solving task for the class, but then, just as they introduce the task at the start of the lesson, it is as though they lose their nerve. They say something like, "Just before you start, let me remind you of a couple of things …". And, in so doing, they destroy the problem that they have carefully constructed, and the lesson becomes an exercise in following the teacher's method. I am sure I have done this myself.

A more subtle way in which the same thing happens is by offering the wrong kind of help to students during the lesson. The teacher is careful not to kill the problem in their lesson introduction by telling the students how to solve it. But, instead, they effectively do the same thing as they circulate around the classroom, making suggestions, giving hints, encouraging the students to take particular approaches. Although this is well meant, I think it can be just as detrimental to students' learning of how to solve mathematical problems. This raises the question of what the teacher's role should be in a problem-solving lesson, which is, for me, the fundamental problem with teaching problem solving. If we teach students how to solve the problem, and then give them the problem, it is not problem solving any more, but, if we just give students the problem, it is not teaching any more.

I have seen problem-solving lessons where the rationale seems to be to give the students an interesting problem to solve and then just stand back and let them struggle with it. I think I have taught lessons like that myself, but I am now unconvinced that that is an effective way of teaching students to solve problems. If the students end up solving the problem without help, then I worry that it was not challenging enough. If they do not, then I worry that they have not learned anything.

How can we make sense of the role of the teacher in a problem-solving lesson? It seems to me that scaffolding the problem is not what we should be doing. Scaffolding a problem inevitably diminishes the problem-solving aspect. It is a short-term aid to solving that particular problem, but it does not future-proof students in preparation for problems that they have not yet met. What we need to be doing is *scaffolding the problem solving,* and this is quite different. When we scaffold the problem, we direct students' attention to particular features of the specific problem that they are working on. When we scaffold the problem solving, we give more generic support, which is intended to help students when they meet other similar problems in the future.

If you take this approach, it means focusing your support, and whole-class discussions, on wider aspects of mathematical problem solving than the particular problem in question and it can be hard for students to appreciate what you are doing. Suppose you pose a problem, such as asking students to "Find all the nets of a cube". They will naturally see this as the objective of the lesson. If you ask a student what they are doing, of course they will say, "I'm trying to find all the nets of a cube." That is what you told them to do. But, for the teacher, the point of the lesson has to be more than that there are 11 nets of a cube and here they are. That is not an important piece of knowledge, certainly not important enough to spend much lesson time ascertaining. It is not worth remembering. The point of the lesson is to learn something about how to solve that kind of problem. Crucial questions for discussion would include:

- How did you go about it?
- How did you check to see if two nets were duplicates of each other?
- What did you count as a duplicate? A reflection? A rotation?
- How can you tackle the problem systematically, so that you know that you must have found them all and can not have missed any?
- How did you go from "this is how many I found" to "this is how many there can be"?
- What other problems have you met that might be solved in a similar way to this one?
- What problems can you invent that could be solved in a similar way to this one?

These are the generic things that can emerge from working on such a problem. But it is easy for students, and their teachers, to overlook them and fixate on solving the particular problem.

Alan Schoenfeld (1985) offers three generic questions to ask students whenever they are stuck on a problem:

- What exactly are you doing? Can you describe it precisely?
- Why are you doing it? How does it fit into the solution?
- How does it help you? What will you do with the outcome when you obtain it?

Note that there is no mathematical content to these questions. Students need to know mathematical content, of course, but that is not the focus here. The goal of the problem-solving lesson is to apply in creative ways the mathematical knowledge that students already have. During the lesson, and afterwards, students should reflect on what worked and what did not, and why, so that they talk not just about the mathematical content but about their approaches to solving the problem.

In working with Japanese colleagues from the IMPULS project (see note at the end of the article), I have learned about the "Japanese problem-solving lesson", where it is said that "the lesson begins when the problem is solved". This means that most of the learning is seen to take place in what Polya (1957, p. 14) called the "looking back" phase, reflecting on the choices made, the paths taken, and the advantages and disadvantages of different approaches. The purpose of time spent solving the problem, or attempting to, is to get students into a position where they have enough relevant experience to contribute to and learn from this critical discussion. I think that that is an interesting way of thinking about the structure of the lesson. The plenary is not an afterthought to tie things together at the end, if there happens to be time. It is the main part of the lesson. What happens before the plenary is seen as preparing students for that discussion.

The great thing about Schoenfeld's three questions is that if the teacher keeps on using them, lesson after lesson, students begin to get to know the questions. The teacher can ask, "What questions am I going to ask you?" and the students will be able to say them. The questions can be put up on the classroom wall, so the teacher can silently point to them. Eventually, the aim is that students internalise these to support their metacognition. They ask these questions of themselves and this helps them to take control of what they are doing.

However, without the necessary toolbox of fluency with facts, procedures and concepts, students will not find these prompts very useful. This is the difficulty with Polya's heuristics, such as that if you cannot solve a problem "try to solve first some related problem" (Polya, 1957, p. 114). This is great, provided that you can tell what a simpler, related problem might be. This is not always easy for a student to do. Making the numbers in a problem smaller does not always make the problem easier. Multiplying by 9 is not easier than multiplying by 10. To take an A-level example, a student trying to solve an integral like $\int x^2\sqrt{1+x^3}\,dx$ might try to use this heuristic by thinking, "This looks complicated. Let's get rid of the x^2 at the front and just solve $\int \sqrt{1+x^3}\,dx$ first, and when I have worked that out I will try it with the x^2 in as well." It seems like a sensible approach. Surely having one factor rather than two to integrate will simplify things? This is a really good problem-solving strategy. But, unfortunately, $\int \sqrt{1+x^3}\,dx$ is a much harder integral, whereas if you think about $\int x^2\sqrt{1+x^3}\,dx$ in the right way, as a "reverse-differentiate" problem, the answer can be written down immediately. The difficulty is, how should the student know what is going to be simpler unless they already understand the structure? If they could see what was going to be simpler, then they would not need to simplify it. So, heuristics like "Solve a simpler problem first" may be a good description of what successful problem solvers do, but they are only helpful to students if they have the necessary background knowledge to make use of them.

Polya recognised all of this, of course. He is credited with saying, "In order to solve this differential equation you look at it till a solution occurs to you" (see http://www-history.mcs.st-andrews.ac.uk/Quotations/Polya.html). If we want solutions or solution methods to occur to our students, they need the necessary rich background knowledge that will enable this to happen. But, it is perfectly possible to have all of the necessary techniques safely inside your toolbox and yet not see how they could help you solve the problem you are tackling. The teacher feels frustrated, because they think that the students ought to be able to solve the problem. They apparently know everything they need to know, but they do not mobilise it in the particular situation they are presented with.

One reason for this may be that the students have met the relevant content only in a narrow range of contexts and have not seen how it might be applied more widely. Another reason may simply be that they have encountered the relevant content too recently. When learning a language, students do not spontaneously and fluently use the vocabulary they have just learned. It needs time to bed in. Similarly, if we want students to make sophisticated use of what they know, it might be better to rely on mathematical content that was learned some time ago and is quite robustly known. Content learned 2 years previously is a rough rule sometimes used at the Shell Centre in Nottingham. This also reduces the tendency for students to assume that the knowledge that they need to solve the problem must be the thing that they have just been taught, thereby switching the task from problem solving to routine exercise. It also acknowledges that if the problem-solving demands are high, other demands, such as procedures and concepts, may need to be lower.

Heuristics like Polya's can be helpful when students have the necessary prior knowledge to solve the problem but a productive approach is not coming into their minds. Then Schoenfeld's three questions can be a powerful way to help students to bring that knowledge to bear on the particular problem they are working on. But, the point is not to solve the problem. Much can be learned even when the problem is not solved. The point is to learn something about solving problems.

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