'excess areas' are the same? What if we try to place several smaller squares in the gap between a square and a circle? How can we construct the largest square inscribed in a triangle or in a regular polygon? How can circles be packed into larger circles? The author provides solutions throughout, many of which share common themes: choose a base length wisely, look for symmetry, see if you can reduce this to something you already know. Moreover, there are plenty of ideas which you, as a teacher, can develop for yourself. I speculated on how the packing problem might be iterated: can an infinite chain of squares, forming a geometric progression, be inserted into a space? And I wondered about generalising some of the exercises: is it really necessary that that square should be fitted symmetrically over the circle for the result to be true?

Subsequent chapters continue this trajectory, and there is far too much on offer to describe everything in a short review. Suffice it to say that in one chapter you will encounter curves of constant width and the Archimedean arbelos, in the next some slicing and folding, including cheesecake, and in another the golden section and Penrose tilings. After this the author generalises to three dimensions, exploring stepped staircases and pyramids, considering both volume and surface area. The final chapter is a collection of intriguing exercises for the reader (with complete solutions).

My enthusiasm for this book should now be clear. It is a splendid illustration of what teachers of the subject ought to offering to their classes, with plenty to engage pupils at all levels of ability. No need to give the bright girl at the back another sheet of examples doing the same thing; just throw a new problem at her. The only thing missing is an index, so if you are going to use this book you are just going to have to read through it. That should, however, be a pleasure.

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Questions pupils ask by Colin Foster, pp. 199, £15.95 (MA members £13.45), ISBN 978-0-90658-891-8, The Mathematical Association (2017).

The author of this collection of pieces from *Mathematics in School* and *Symmetry Plus* claims that it is through *asking questions* that professional mathematicians explore and develop their subject. Similarly, in a school context, teachers have a duty to show respect to pupils who ask questions and to provide them with helpful answers which enhance their understanding.

Some years ago I delivered a series of geometry masterclasses to Y10 and Y11 pupils at a summer school in Oxford. The participants were encouraged both to ask and answer questions. At the end of the week we had a feedback session, and I asked my group to tell me how their experience at the summer school differed from that in the classroom. One of the boys responded as follows:

> "When I ask my teacher at school why something he has taught us is true, he answers 'Because I told you so!'. Here, when we ask you a question, you help us work out the answer for ourselves. When we offer an answer to a question you have asked, you challenge us to justify the details."

Obviously this is an extreme situation, but I am sure that Colin Foster would appreciate what the pupil was saying. Throughout this short book he gives us example after example of the dialogue between teacher and learner which is essential to the development of mathematical ability, and he makes it clear that even the most banal question can lead to an interesting exchange of ideas and a growth in confidence and enjoyment – for both pupil and teacher. Teachers, he claims, have a lot to learn from their pupils' questions, and their own understanding of basic concepts will grow accordingly.

The first part of the book is devoted precisely to these questions, and, as expected, the process of providing answers is not simply to say that this is how things are. As an example, the first chapter asks, 'If two minuses make a plus, why don't two pluses make a minus?' What was meant by 'two minuses make a plus'? Does it mean that $3 - (-4) = 3 + 4$ or that $(-3) \times (-4) = +12$? These are clearly very different processes, and it is the teacher's job to elucidate why each statement turns out to be true, not to rely on a mantra which might help weaker pupils to do the right thing but is actually a barrier to understanding. I am not sure that I agree with the author's own explanation, which tends to rely on an analogy with electrostatic charges, but I certainly think it is important to talk about it. My own approach would certainly distinguish between the 'minus sign' as an operation on numbers and as a descriptor of negative numbers, and would probably evoke the image of the number line to show why $3 - (-4) = 3 + 4$. To explain why $(-3) \times (-4) = +12$, I would treat multiplication as repeated addition, talk about credits and debits, and ask: if I gain three units every week, what is my situation in four weeks time, and what was my situation four weeks ago? what would happen if I lost three credits every week? To each his own, of course, and maybe your actual approach does not matter, so long as it is honest, rigorous and robust.

A further question is: why did the pupil ask this question? Did it actually show a perception that mathematics is about symmetry? Perhaps the question was not as stupid as it seems. Other ideas examined in this section include: why is 1 not a prime number? how do we know that π is irrational? is there a formula for factorials (like the formula for the sum of the first n natural numbers? is there a net for a parallelepiped?

The second part of the book is 'reflections on topics', and here it is the teacher who is asking the questions. What is the best way to teach difference tables? Why do we use inclusive definitions (such as the fact that every rectangle is a parallelogram) sometimes, and exclusive definitions (such as a scalene triangle not being isosceles) at other times? When are the different statistical measures of central tendency – mean, median, mode – useful? How many factors does a positive integer have? There are some interesting ideas here and I think most teachers will be stimulated to reflect on their own practice.

The final part is a series of problems, puzzles and tasks. I found this the least interesting part of the book, probably because I already have many of these in my repertoire already. However, I enjoyed a discussion of a comment made by David Attenborough about the use of food porters in expeditions into remote areas. Obviously any such bearers have to feed themselves as well as those in the party who are not carrying provisions. Two carriers and three eaters can manage reasonably comfortably for 14 days, but if the expedition becomes any longer then more porters are needed. At what point does the length and the size of an expedition become impractical? It is sooner than you might think.

This whole book is testimony to the fact that it is much better to teach mathematics by allowing ideas to develop than to present it as a series of algorithms, and I confess to being particularly encouraged by a comment at the end of an excellent chapter on gradients:

> "Unfortunately, the tendency or requirement to state learning objectives precisely at the beginning of a lesson can make it difficult for concepts to emerge naturally in a problem-solving setting like this."

I cannot agree more with this sentiment.

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Coming to know: the introduction of new concepts in undergraduate mathematics by R. P. Burn, pp. 114, £8.59, ISBN 978-0-955-59142-6, MSOR Network (available via lulu.com) (2009).

This textbook consists of roughly eighty worksheets on a range of topics that one might encounter in the first half of a typical undergraduate mathematics degree. Most of the worksheets take up exactly one full page of fairly large text, all of which is helpfully ringbound for easy reference. The worksheets are almost entirely independent, though each one lists some basic pre-requisites. However, the aim of each is to provide the context and build towards a topic, rather than give details or technical definitions within the topic itself.

So, for example worksheet 5.10, entitled *Preparing for the Cayley-Hamilton Theorem*, has the reader calculate some 2×2 matrix products and discover the minimal polynomial by inspection in some simple examples, and encourages the student to compare with the characteristic polynomial. It finishes by asking for comments on extrapolating to higher dimensions, and providing some historical context from Cayley himself concerning his soon-to-be-published theorem.

The author explains his pedagogical philosophy at some length. Inevitably, research mathematics, he says, aims to 'allow the verification of formal accuracy by the reader' and this convention of succinct exposition makes learning 'artificially difficult for undergraduates' when used without enough motivation and context. While the introduction and especially the sources cited might not stand up as a totally unbiased review of the situation, I'm sure anyone who has attended an undergraduate course or read an undergraduate textbook can agree that the point is compelling and somewhat representative.

A counterargument is fairly obvious, namely that exploring and deciding on the context, parsing new definitions, and testing them on small examples, to some extent *is* what it means to learn mathematics. At some point between high school and research, one has to make this transition. But undergraduate mathematics courses do not exist solely to prepare future research mathematicians, and so it is reasonable to imagine that it might be of considerable long-term benefit to ease this transition, and, if not actually delay it, at least stretch it out over a longer period, with explicit guidance that this is what is happening.

To my taste, some of the worksheets are rather basic for undergraduate level. While worksheet 1.3 is intended as an introduction to proof, there are probably less banal examples than that the sum of two even integers is even. However, others might be excellent for challenging school students at GCSE or A-level. One might assume that an undergraduate studying graph theory, seeing Euler's theorem on faces, edges and vertices for the first time, might experiment with some small connected planar graphs (and perhaps even some small non-connected planar graphs?) without such explicit prompting. However, the presentation of the graph-based inductive proof would I imagine be both accessible and interesting to a less-experienced student.