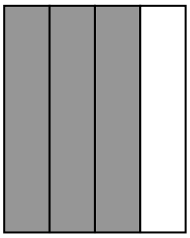
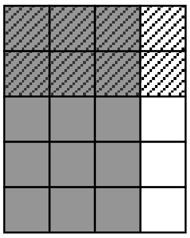
1.8 Fractions (Multiplication and Division)

- Multiplication of fractions is an easier *process* than addition and subtraction, although harder conceptually. We can just "define" multiplication of fractions as "multiply the tops, multiply the bottoms", but pupils need to see that this is a sensible definition in the context of integers. (The principle is that you can define anything you like in maths so long as it's useful and doesn't contradict anything you've already defined.)
- Area provides a useful context: a room $2\frac{1}{2}$ m by 3 m will clearly have an area $1\frac{1}{2}$ m^2 bigger than a room only 2 m by 3 m. (A drawing makes this clear.)
- To see that $\frac{2}{5}\times\frac{3}{4}=\frac{3}{10}$, draw a rectangle and divide it into 4 equal *vertical* strips. Shade 3.



This represents $\frac{3}{4}$ of the total area.

Then divide the whole rectangle into 5 equal *horizontal* strips. Shade 2 of them.



Where the shadings overlap there are 6 twentieths of the whole rectangle; and you can see that this is the same as $\frac{3}{10}$. So $\frac{2}{5}$ of $\frac{3}{4}$ is $\frac{6}{20}$, or $\frac{3}{10}$.

- It's important to make a habit of cancelling as much as possible before multiplying so that you have easier multiplications to do and less simplifying at the end (none if you cancel as much as possible).
- For mixed problems involving + – × ÷ , BIDMAS becomes essential.

1.8.1	A possible context for multiplication would be winning £$\frac{3}{4}$m in a competition and wanting to share it equally between yourself and a friend.	*Answer:* $\frac{1}{2}$ *of* £$\frac{3}{4}$m is £$\frac{3}{8}$m = £375 k "$\frac{1}{2}$ *of" is the same as "* ÷ *2"*
1.8.2	A possible context for division by a fraction could be making a rice pudding. It takes $\frac{3}{4}$ of a pint of milk to make a rice pudding. How many rice puddings can I make with 18 pints of milk? Pupils could discuss in groups what they think the answer is and why. *Pupils can check afterwards that calculators agree.*	*Answer: It should be clear that the question is how many* $\frac{3}{4}$ *'s go into 18, so we need to work out* $18\div\frac{3}{4}$*, and the answer is 24.* *One approach is to see that* $18\div\frac{1}{4}=18\times 4$ *(how many quarter pints can you get out of 18 pints) and that* $18\div\frac{3}{4}$ *will be one third of this amount.* *Another approach is to view division as repeated subtraction and see how many times you can take away* $\frac{3}{4}$ *from 18 before all the milk has gone.*
1.8.3	Puzzle pictures (colour in the answers to produce a picture).	*Often popular and available in books.*

1.8.4 Sometimes you can divide fractions without needing the "turn upside down and multiply" rule; e.g., it can be seen that $\frac{3}{5}\div\frac{4}{5}=\frac{3}{4}$, because "3 anythings, divided by 4 of the same must make $\frac{3}{4}$".
Some *divisions* can be done fairly easily in this way by finding *common denominators*; e.g., $\frac{2}{3}\div\frac{4}{5}=\frac{10}{15}\div\frac{12}{15}=\frac{10}{12}=\frac{5}{6}$.

This is a good opportunity to see that there is frequently more than one way of solving a problem, and that different methods are better in different circumstances.

This may be longer but makes more sense to some pupils, although it would lead to unnecessary work if you had, say, $\frac{1}{19}\div\frac{1}{20}$.

1.8.5 It's helpful to see some patterns in multiplying and dividing different fractions (see "Multiplying and Dividing Fractions" sheet).

Explaining why the patterns (e.g., $\frac{1}{3}$ times table) come where they do is challenging.

This helps to see that the answers fit in with the answers to multiplying and dividing integers. This reinforces the idea that fractions are just numbers.

Not every one has to be worked out once the patterns are spotted.

1.8.6 Work out $(1+\frac{1}{2})(1+\frac{1}{3})(1+\frac{1}{4})(1+\frac{1}{5})$.

Answer:
$$\frac{\not{3}}{2}\times\frac{\not{4}}{\not{3}}\times\frac{\not{5}}{\not{4}}\times\frac{6}{\not{5}}=\frac{6}{2}=3,$$
cancelling before multiplying.

1.8.7 A spider climbs up out of a bath 50 cm tall. Each day he climbs up $\frac{2}{3}$ cm, and each night he slips back $\frac{1}{4}$ cm. If he starts at 6 am (the beginning of the day) on the first day, on which day will he make it completely out of the bath?

If the spider starts at the beginning of day 1 (say 6 am) and climbs at a steady rate during the days (still slipping back at night) at what time on the final day will he make it over the side of the bath?

Answer: day 120. After each day and night he ends up $\frac{2}{3}-\frac{1}{4}=\frac{5}{12}$ cm further up the bath, so $50\div\frac{5}{12}=120$, but this includes slipping back at the end of the 120th day.
So to be more accurate, at the end of the 119th day he will have achieved $119\times\frac{5}{12}=49\frac{7}{12}$ cm, so he has only $\frac{5}{12}$ cm to go. This will take $\frac{5}{12}\div\frac{2}{3}$ of a day $=\frac{5}{8}$ of a day $=\frac{5}{8}\times 12$ hours through the day $=7\frac{1}{2}$ hours after 6 am = 1.30 pm on the 120th day.

1.8.8 Pupils can "invent" rules for multiplying and dividing fractions by trying to make their behaviour fit with what they know happens with decimals.
e.g., $0.5\times 0.5=0.25$ according to knowledge / calculator / "boxes method" / whatever.
What rule can we use with $\frac{1}{2}\times\frac{1}{2}$ to make it give us $\frac{1}{4}$?

A calculator that will not do fractions may be useful.
(Decimals are "decimal fractions" anyway.)

It doesn't matter if pupils don't recognise what fraction is equivalent to the decimal answers they obtain: they can experiment with different fractions to see which turns out to be equivalent (numerator ÷ denominator on the calculator).

1.8.9 There is a natural link with probability work: When should you multiply probabilities and when can't you?

Probabilities are usually written as fractions or decimals.

You always *multiply probabilities along the branches of a tree diagram, because the probability written on any branch is always* conditional *on the necessary events happening along the branches that lead to that branch. (See section 3.5 for more on this.)*

Answer:
When you want the probability of two events A and B both happening, you multiply:
$p(A\cap B)=p(A)p(B|A)=p(B)p(A|B)$, *where $p(A|B)$ means the conditional probability that A happens given that B has already happened.*
If A and B are independent events – i.e., the chance of A happening is the same whether or not B happens – then these formulas reduce to $p(A\cap B)=p(A)p(B)$, because $p(B|A)=p(B|\bar{A})=p(B)$, etc.

Multiplying and Dividing Fractions

Fill in the missing numbers in these grids. Write all the answers as mixed numbers with fractions in their lowest forms.

(horizontal number *multiplied* by vertical number)

×	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5
$\frac{1}{2}$	$\frac{1}{4}$									
1			$1\frac{1}{2}$							
$1\frac{1}{2}$										
2										
$2\frac{1}{2}$										
3										
$3\frac{1}{2}$										
4										
$4\frac{1}{2}$										
5										

(*horizontal* number *divided* by *vertical* number each time)

÷	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5
$\frac{1}{2}$	1	2								
1	$\frac{1}{2}$									
$1\frac{1}{2}$	$\frac{1}{3}$									
2										
$2\frac{1}{2}$										
3										
$3\frac{1}{2}$										
4										
$4\frac{1}{2}$										
5										

Extra Task Make up a number square like this with the fractions around the edge going up in $\frac{1}{3}$'s or $\frac{1}{4}$'s or some other fraction.

Multiplying and Dividing Fractions **ANSWERS**

Fill in the missing numbers in these grids. Write all the answers as mixed numbers with fractions in their lowest forms.

(horizontal number *multiplied* by vertical number)

$\times$	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5
$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$1\frac{1}{4}$	$1\frac{1}{2}$	$1\frac{3}{4}$	2	$2\frac{1}{4}$	$2\frac{1}{2}$
1	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5
$1\frac{1}{2}$	$\frac{3}{4}$	$1\frac{1}{2}$	$2\frac{1}{4}$	3	$3\frac{3}{4}$	$4\frac{1}{2}$	$5\frac{1}{4}$	6	$6\frac{3}{4}$	$7\frac{1}{2}$
2	1	2	3	4	5	6	7	8	9	10
$2\frac{1}{2}$	$1\frac{1}{4}$	$2\frac{1}{2}$	$3\frac{3}{4}$	5	$6\frac{1}{4}$	$7\frac{1}{2}$	$8\frac{3}{4}$	10	$11\frac{1}{4}$	$12\frac{1}{2}$
3	$1\frac{1}{2}$	3	$4\frac{1}{2}$	6	$7\frac{1}{2}$	9	$10\frac{1}{2}$	12	$13\frac{1}{2}$	15
$3\frac{1}{2}$	$1\frac{3}{4}$	$3\frac{1}{2}$	$5\frac{1}{4}$	7	$8\frac{3}{4}$	$10\frac{1}{2}$	$12\frac{1}{4}$	14	$15\frac{3}{4}$	$17\frac{1}{2}$
4	2	4	6	8	10	12	14	16	18	20
$4\frac{1}{2}$	$2\frac{1}{4}$	$4\frac{1}{2}$	$6\frac{3}{4}$	9	$11\frac{1}{4}$	$13\frac{1}{2}$	$15\frac{3}{4}$	18	$20\frac{1}{4}$	$22\frac{1}{2}$
5	$2\frac{1}{2}$	5	$7\frac{1}{2}$	10	$12\frac{1}{2}$	15	$17\frac{1}{2}$	20	$22\frac{1}{2}$	25

(*horizontal* number *divided* by *vertical* number each time)

$\div$	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5
$\frac{1}{2}$	1	2	3	4	5	6	7	8	9	10
1	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5
$1\frac{1}{2}$	$\frac{1}{3}$	$\frac{2}{3}$	1	$1\frac{1}{3}$	$1\frac{2}{3}$	2	$2\frac{1}{3}$	$2\frac{2}{3}$	3	$3\frac{1}{3}$
2	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$1\frac{1}{4}$	$1\frac{1}{2}$	$1\frac{3}{4}$	2	$2\frac{1}{4}$	$2\frac{1}{2}$
$2\frac{1}{2}$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	1	$1\frac{1}{5}$	$1\frac{2}{5}$	$1\frac{3}{5}$	$1\frac{4}{5}$	2
3	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	1	$1\frac{1}{6}$	$1\frac{1}{3}$	$1\frac{1}{2}$	$1\frac{2}{3}$
$3\frac{1}{2}$	$\frac{1}{7}$	$\frac{2}{7}$	$\frac{3}{7}$	$\frac{4}{7}$	$\frac{5}{7}$	$\frac{6}{7}$	1	$1\frac{1}{7}$	$1\frac{2}{7}$	$1\frac{3}{7}$
4	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{5}{8}$	$\frac{3}{4}$	$\frac{7}{8}$	1	$1\frac{1}{8}$	$1\frac{1}{4}$
$4\frac{1}{2}$	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{1}{3}$	$\frac{4}{9}$	$\frac{5}{9}$	$\frac{2}{3}$	$\frac{7}{9}$	$\frac{8}{9}$	1	$1\frac{1}{9}$
5	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{3}{10}$	$\frac{2}{5}$	$\frac{1}{2}$	$\frac{3}{5}$	$\frac{7}{10}$	$\frac{4}{5}$	$\frac{9}{10}$	1