1.14 Indices

- The rules of indices can be introduced with powers of ten. 10^3 is $10 \times 10 \times 10$ and 10^5 is $10 \times 10 \times 10 \times 10 \times 10$, so multiplying these gives $10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 = 10^8$. Multiplying an integer by 10 means adding a 0 on to the right end, so adding 1 to the power of 10 makes sense. The division rule comes from cancelling tens from the top and bottom of a fraction, and when the denominator is larger than the numerator we're in to negative indices.

  Alternatively, multiplying by 10^a moves all the digits a places to the left, and dividing by 10^b moves all the digits b places to the right, so the overall effect is a move $a-b$ places to the right (or $b-a$ places to the left); so $10^a \div 10^b = 10^{a-b}$.
- The trickiest rule is $x^{\frac{a}{b}} = \sqrt[b]{x^a}$, and this is probably better done (especially initially) in two stages, so e.g., $8^{\frac{2}{3}} = (8^{\frac{1}{3}})^2 = (\sqrt[3]{8})^2 = 2^2 = 4$.

  Confident calculator use is very important, using the x^y or y^x or ^ buttons for powers, but the EXP button for standard form (and *not* typing in the 10 or the ×); i.e., for 6×10^4 typing "6 EXP 4" only.

1.14.1 "Sarah, give me a number between 1 and 10." Sarah says (say) 6; I write 36 on the board. Repeat a couple of times. "What's going on?"

Treating the teacher as a function machine. Can use for "cubing", etc. A good way of getting pupils to express in words what a function is doing to a number.

1.14.2 What's the difference between 3^4 and 4^3? Which do you think is bigger? Can you think of cases where the answers are the same? i.e., $a^b = b^a$

Explain the difference between 3^4 and 4×3.

$3^4 = 3 \times 3 \times 3 \times 3$, and $4 \times 3 = 3 + 3 + 3 + 3$ ($4 \times 3 = 3 \times 4 = 4 + 4 + 4$)

$3^4 = 3 \times 3 \times 3 \times 3 = 81$; $4^3 = 4 \times 4 \times 4 = 64$

Obviously, there are solutions if $a = b \neq 0$ (0^0 is undefined); e.g., $3^3 = 3^3$. There are also infinitely many solutions where $a \neq b$: one value is always between 1 and e (2.71828...) and the other is $> e$. The only integer solution where $a \neq b$ is $2^4 = 4^2$.

1.14.3 Calculating square roots by hand may be of interest to some pupils.

See old-fashioned textbooks or the internet.

1.14.4 Work out 4^1, 4^2, 4^3 and 4^4. What are the units digits for each one? What do you notice? What would the last digit be in 4^{20}? What about 4^{123}?

Answers: 4, 6, 4, 6

The units digit in 4^n is 4 when n is odd and 6 when n is even. This works only for positive integer values of n. For $n = 20$, the last digit will be 6; for $n = 123$ it will be 4.

1.14.5 Are these always true, sometimes true (if so say when) or never true? (Try it with numbers to see.)

1. $(a+b)^2 = a^2 + b^2$
2. $(a-b)^2 = a^2 - b^2$
3. $(ab)^2 = a^2 b^2$
4. $(\frac{a}{b})^2 = \frac{a^2}{b^2}$
5. $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$
6. $\sqrt{a-b} = \sqrt{a} - \sqrt{b}$
7. $\sqrt{ab} = \sqrt{a}\sqrt{b}$
8. $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

Answers: ("iff" means "if and only if")

1. *true iff a or $b = 0$*
2. *true iff $b = 0$ or $a = b$*
3. *true always*
4. *true unless $b = 0$*
5. *true iff a or $b = 0$*
6. *true if $b = 0$ and $a > 0$ or $a = b > 0$*
7. *true unless a or $b < 0$*
8. *true unless $a < 0$ or $b \leq 0$*

1.14.6 Find two consecutive integers whose squares differ

Answer: 15 and 16

by 31.
Find two consecutive odd numbers whose squares differ by 80.

Answer: 19 and 21

1.14.7 Sum consecutive square numbers and try to verify that $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$.

Could also try $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1) = \left(\sum_{r=1}^{n} r\right)^2$.

And even the sums of 4th and 5th powers:

$$\sum_{r=1}^{n} r^4 = \frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1)$$

$$\sum_{r=1}^{n} r^5 = \frac{1}{12}n^2(n+1)^2(2n^2+2n-1)$$

1.14.8 There are 20 integers less than 30 that can be written as the difference of two squares.
What are they?

3, 5, 7, 8, 9, 11, 12, 13, 15 (2 ways), 16, 17, 19, 20, 21 (2 ways), 23, 24 (2 ways), 25, 27 (2 ways), 28, 29
The only ones that won't work are 2 × an odd number.

1.14.9 Square and square root as inverses.
I think of a number and square it. I get 36. The number wasn't six. What was the number?
Does this happen with cubes as well?

Answer: –6

Answer: No, because $-6\times-6\times-6=-216$.

Could try to find $\sqrt{-36}$.
Why does the calculator say "ERROR"?

There's no "real number answer", but there are so-called "imaginary" or "complex" numbers. The answer to this would be $6i$ *, where* $i=\sqrt{-1}$.

1.14.10 Happy Numbers
"Take a 2-digit number. Square each digit and add up the numbers you get (you could call it the "square digit sum"). Keep going until you have a good reason to stop."
If you hit a 3-digit number, just square each digit and add the three numbers you get, and carry on.

It's called a "Happy Number" if you end up at 1 eventually when you do this.
You can make a "map" of which numbers lead to which numbers and look for loops and branches.
Happy numbers include 10, 100, 1000 (1 step away from 1); 13, 31, 130, 86, 68 (2 steps away from 1) and 28, 82, 32, 23, 79, 97 (3 steps away from 1).

1.14.11 Squares on a Chessboard.
How many squares are there on a chessboard? There are more than 64.
Start small (i.e., a 1 × 1 board has just 1 square). Be systematic.

Answer: 204 (fewer than you might have guessed); the sum of the 1st eight square numbers. Count all of size 1 × 1 (8 × 8 = 64), size 2 × 2 (7 × 7 = 49), etc., separately and add them up. Or find the total for different sized boards (1, 5, 14, 30, 55, 91, 140, 204).

Extension: How many rectangles are there?
(A square counts as a rectangle.)

Answer: 1296 (the sum of the 1st eight cube numbers) (see section 1.17.2).

1.14.12 Towers of Hanoi.
Ancient puzzle: You have 3 towers, n discs stacked on one of them in order of size (biggest at the bottom). You have to get the whole stack onto another tower in the same order (biggest at the bottom). You can only move one disc at a time, and at no point are you allowed to have a larger disc on top of a smaller disc.
What is the minimum number of moves possible?

$m_n = 2m_{n-1}+1$ *is an inductive formula.*

Answer: 2^n-1*, a big number if you have more than a few discs.*
One way to think of it is mentally to split up a stack of n *discs into the one on the bottom and the stack of* $n-1$ *discs above it. If* m_{n-1} *is the number of moves for* $n-1$ *discs, then you just have to move them onto the middle stack and then move the one remaining disc onto the third stack. Then move the* $n-1$ *discs again onto the third stack (another* m_{n-1} *moves) on top of the biggest disc and you're done.*
Therefore $m_n = m_{n-1}+1+m_{n-1} = 2m_{n-1}+1$.
Since you're doubling (and adding one) each time, the formula is bound to involve 2^n.

1.14.13 Sweets (or grains of rice on a chessboard).
"In the old days in maths lessons, pupils used to sit with the person who was the best right at the front, then the next best person next to them, and so on till the person who was the worst right at the back. (I

Careful who's sitting where before you do this!

It's surprising how much pupils will enjoy an imaginary and unfair distribution of sweets!

don't know how they did it because Katie might be better than David at one bit of maths and David might be better than Katie at another bit of maths, so it can't have been easy.) Imagine we were sat like that today, and I decided to give out sweets. David (at the back) gets 1 sweet (to console him), Jenny gets 2 sweets, Henry gets 4 sweets, and so on, doubling each time.
I have two questions.
First, how many sweets will Katie (in number 1 position) get, and second, how many sweets would I need to buy to do this."

Imagine a class of size 10 and see what happens. Pupils may think "proportionally" (e.g., "work it out for the first 4 pupils and times it by 8 for all 32 pupils"), but it doesn't work.

"Katie" will get 2^{n-1} sweets, where n is the class size today; e.g., for $n = 30$, $2^{29} = 536{,}870{,}912$.

Total number of sweets = $\sum_{i=1}^{n} 2^{i-1} = 2^n - 1$.

So each pupil gets 1 more than the total of everyone up to them.
So for a class of 30 I'd need 1,073,741,823 sweets (over a billion). If they were penny sweets, it would cost me £10,737,418.23.

How many years of my salary (say £20 000 per annum) would it cost?

About 500 years (assuming I spent no money on anything else)!

1.14.14 Tournaments.
Imagine we all want to play in a table tennis tournament. What if there were 32 people here?

Answers:
Works nicely with a power of 2: 16 matches in 1st round, 8 in 2nd, 4 in 3rd, 2 semi-finals and 1 final (draw a tree diagram).

How many matches would there be?

31 (= 32 – 1) because $\sum_{i=1}^{n} 2^{i-1} = 2^n - 1$.

What if Edward didn't want to play and we couldn't find anyone else?
Would we have to cancel the whole thing?

No – someone could join in at round 2 and then it would be OK.
The number of matches is still 1 less than the number of people playing because if n people take part then $n-1$ people must lose once each (the other person wins the tournament). Each match has one winner and one loser, so there must be $n-1$ matches.

1.14.15 I have 2 parents, 4 grandparents, 8 great-grandparents, and so on. If I count parents as 1 generation back, grandparents as 2 generations back, etc., then how many great-great-etc.-grandparents do I have 20 generations back?
Estimate how long ago that would have been.

Answer: 2^{20} = 1 048 576 people 20 generations back, and taking 25 years as an average "generation gap" this would have been 20 × 25 = 500 years ago (Elizabethan times).

What about 30 generations back?

Again, 2^{30} = about 1 billion ancestors 30 × 25 = 750 years ago (medieval times).

What about 33 generations back?

Now 2^{33} = 8.6 billion, more than the total number of people in the world today, and there were far fewer then, and these are just my *ancestors! What about everyone else's?*

It's estimated that 1000 years ago there were only about 300 million people in the world in total! If everyone is descended from "Adam and Eve", then we must share a lot of distant ancestors.

The answer is that we're counting the same people more than once because some of my ancestors will have married people with some of the same ancestors.

1.14.16 Knights of the Round Table (see sheet; answers below).

Interesting problem involving powers of 2.

1.14.17 Why do the square numbers alternate odd, even, odd, even, …?

Answer: Odd number × odd number = odd number, and even × even = even.
So because the positive integers alternate odd, even, odd, even, so do the square numbers.

1.14.18	Exponential Curves (growth or decay). e.g., bacteria growth curves (numbers of bacteria against time); rabbit population in Australia (numbers of rabbits against time); radioactive decay curves (activity against time), cooling curves (temperature against time).	*An opportunity for some mathematical modelling and discussing of assumptions.* *In each case, the rate of increase or decrease is roughly proportional to the current value.*
1.14.19	The number of bacteria in my fridge doubles every week. If there were 20 000 at the end of the 12th week, after how many weeks were there 10 000 bacteria?	*Answer: 11 weeks (obvious when you think about it).*
1.14.20	Which do you think is larger: $2\sqrt{\frac{2}{3}}$ or $\sqrt{2\frac{2}{3}}$? Find another instance of this.	*Answer: they're equal; easily seen by squaring the left hand side to give* $4\times\frac{2}{3}=2\frac{2}{3}$. *Other examples are* $3\sqrt{\frac{3}{8}}=\sqrt{3\frac{3}{8}}$, $4\sqrt{\frac{4}{15}}=\sqrt{4\frac{4}{15}}$, $5\sqrt{\frac{5}{24}}=\sqrt{5\frac{5}{24}}$, *etc.*
1.14.21	Logarithms. Pupils could experiment with the mysterious $\log$ button on the calculator to try to work out what it does. Pupils could be asked to work out $\log 1000$, $\log 100$, $\log 1$, $\log 0.01$ and to try to relate the answers to the numbers that were "logged". Why do $\log 0$ and $\log -10$ give ERROR?	*Answers: 3, 2, 0, –2* *Logarithms are just powers; the logarithm of a number is the power that 10 would have to be raised to to get that number, so because* $10^{-2}=0.01$ *we can say that* $-2=\log 0.01$. *(Logarithms can be to* bases *other than 10.)* *The equations* $10^x=0$ *and* $10^x=-10$ *have no real solutions.*

Knights of the Round Table ANSWERS

The only safe position is 73 places from the left of the King.

See the table on the sheet for the best position p for different numbers of knights n. Positions are counted clockwise around the table, counting as 1 the knight immediately to the King's left. Every time n reaches a power of 2 (e.g., $16 = 2^4$), the winning position goes back to $p=1$, and from then onwards the presence of every additional knight moves the optimum position two knights further on.

So if $m=$ the biggest power of 2 that is less than n, then $p=2(n-m)+1$.

e.g., if $n=100$, then $m=2^6=64$, and therefore $p=2(100-64)+1=73$.

So for 100 knights the 73rd position is the best.

And if $n=1000$, $m=2^9=512$, and therefore $p=2(1000-512)+1=977$.

So for 1000 knights the best position is the 977th.

Knights of the Round Table

Once upon a time there was a King who had a round table.

He wanted his daughter to get married, and he invented an unusual way of deciding which of his knights she would marry.

He invited the knights to dinner and sat them at the round table.

After dinner the King took out his sword, and passing by the knight immediately to his left, cut off the head of the knight next to him.

He then went on around the table cutting off the head of every other knight.

He kept going – missing himself out, of course! – until there was just one knight left.

This was the knight who would survive to marry the King's daughter.

So if 100 knights are invited, where is the best place to sit?

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Knights of the Round Table

n = number of knights at the table ***p*** = position of the knight who wins

n	*p*	*n*	*p*	*n*	*p*	*n*	*p*
1	1	26	21	51	39	76	25
2	1	27	23	52	41	77	27
3	3	28	25	53	43	78	29
4	1	29	27	54	45	79	31
5	3	30	29	55	47	80	33
6	5	31	31	56	49	81	35
7	7	32	1	57	51	82	37
8	1	33	3	58	53	83	39
9	3	34	5	59	55	84	41
10	5	35	7	60	57	85	43
11	7	36	9	61	59	86	45
12	9	37	11	62	61	87	47
13	11	38	13	63	63	88	49
14	13	39	15	64	1	89	51
15	15	40	17	65	3	90	53
16	1	41	19	66	5	91	55
17	3	42	21	67	7	92	57
18	5	43	23	68	9	93	59
19	7	44	25	69	11	94	61
20	9	45	27	70	13	95	63
21	11	46	29	71	15	96	65
22	13	47	31	72	17	97	67
23	15	48	33	73	19	98	69
24	17	49	35	74	21	99	71
25	19	50	37	75	23	100	73