1.17 Triangle Numbers

- The n^{th} triangle number is $\frac{1}{2}n(n+1)$. The first few are 1, 3, 6, 10, 15, 21, …
  The difference between each number and the next goes up by 1 each time.
  The formula $\frac{1}{2}n(n-1)$ gives the $(n-1)^{\text{th}}$ triangle number for $n \geq 2$.
  The n^{th} triangle number is the same as ${}^{n+1}C_2$.
- Triangle numbers can be illustrated with equilateral or right-angled isosceles triangles of dots.
  We include 1 because it completes the pattern of differences and fits the above formulas.
  For the same reason, we say that 1 is the first *square number*, since 1 × 1 = 1.

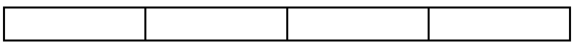
1.17.1 Handshakes.
If everyone in the room were to shake hands with everyone else in the room, how many handshakes would there be?
Start small. Try 3 people. Stand up and do it.
Now try a group of 4. Somebody keep count.

If there are n people, each of them needs to shake hands with everyone except themselves; i.e., $n-1$ people, so that makes $n(n-1)$, but this counts every handshake twice "from both ends". So the answer is $\frac{1}{2}n(n-1)$.

An alternative context is sending "Christmas cards" (or similar) – if everyone sends one to everyone else how many cards will there be?
This avoids the problem of double-counting, since if A sends a card to B, B (out of politeness!) will send a card to A.
You can do this with teams playing matches if you say that every team wants to play every other team "*at home*".

Each person has to send $n-1$ cards, because they send one to everybody except themselves. There are n people that do that, so $n(n-1)$ cards altogether.
If n teams all play each other "at home", then all n venues are visited by all $n-1$ other teams (obviously they don't play against themselves!) and that is just $n(n-1)$ games.

1.17.2 Rectangles. How many rectangles (of any size) are there in this shape?
There are more than 4.



Answer: 10, the 4th triangle number

You can explain it by thinking about what happens when you add another block on to the right end to make 5 blocks. The 5th block makes 5 more rectangles: itself, itself and the one to its left, itself and the two to its left, itself and the three to it's left, and itself and the four to its left. This happened because there were already 4 rectangles there.
In general, adding the n^{th} rectangle increases the total by n.

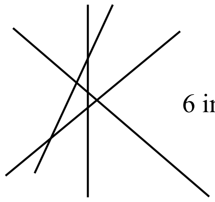
Can extend to 2 dimensions (i.e., like a chessboard).
Here, a square counts as a rectangle.

An array of rectangles $x \times y$ will contain a total of $\frac{1}{2}x(x+1) \times \frac{1}{2}y(y+1)$ rectangles $= \frac{1}{4}xy(x+1)(y+1)$. This is because the first row contains $\frac{1}{2}x(x+1)$ different rectangles, and each of these has height 1, so $\frac{1}{2}y(y+1)$ of them can be fitted vertically down the grid.
If $x = y$ (a square array, like a chessboard), this reduces to $= \frac{1}{4}x^2(x+1)^2$, so when $x = 8$ (for a chessboard), the total is 1296 rectangles (the sum of the 1st eight cube numbers).
(See section 1.14.11.)

1.17.3 Straight lines and intersections. How many crossing-points are there when 4 lines overlap if each new line

Need to use A4 paper at least and choose the angles of the lines wisely.

is drawn so that it crosses as many lines as possible?



6 intersections

How many regions are produced at each stage? (Count 1 line as producing 2 regions.)
A context for this is cutting up a cake:
"What is the maximum number of pieces you can divide a cake into using 5 straight cuts? The pieces don't have to be equal sizes."
Start small – try 1, 2, 3 cuts first.

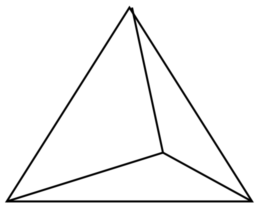
It's easy to miscount or draw lines which don't cross the maximum possible number of lines.

The n^{th} line should cross $n-1$ lines, so after the n^{th} line there should be $\frac{1}{2}n(n-1)$ intersections. (Every line crosses $n-1$ others, making $n(n-1)$ crossings, but this double-counts every line, so we put in a factor of $\frac{1}{2}$.)
Or you can say that there will be a crossing-point for each pair of lines, so it's the number of ways of choosing 2 from n , or ${}^nC_2 = \frac{n(n-1)}{2!}$, which is the same.

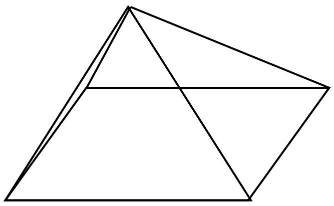
regions = pieces of cake after n cuts = $\frac{1}{2}(n^2+n+2) = \frac{1}{2}n(n+1)+1$; i.e., one more than the n^{th} triangle number.

See a similar investigation (about the maximum number of triangles *formed from intersecting lines) in section 1.19.12.*

1.17.4 Baked Bean Tins.
These can be stacked in a triangular array, giving the triangle numbers. This is easy to extend to *tetrahedron numbers* by making a triangular horizontal layer and putting a smaller triangular layer on top until you reach one can at the very top. The finished stack is roughly tetrahedral.



Making each layer a *square* array of cans instead, gives the *square pyramidal numbers*.



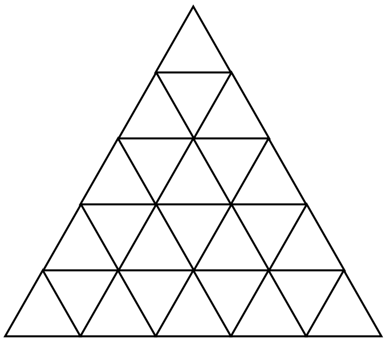
*The n^{th} **tetrahedron number** is the sum of the first n triangle numbers, so they go 1, 4, 10, 20, 35, 56, 84, ...; in general the n^{th} one is $\frac{1}{6}n(n+1)(n+2)$.*

Why does this formula always give a positive integer whenever n is a positive integer?
Answer: Because n , $n+1$ and $n+2$ are consecutive integers. This means that either one of them or two of them must be even, so when you multiply all three you're bound to get a number that is divisible by 2. Also, one of them will always be a multiple of 3 (because multiples of 3 come every three numbers), so the product will be divisible by 3. So when you divide by 6 you get an integer.

*The n^{th} **square pyramidal number** is the sum of the first n square numbers, so they go 1, 5, 14, 30, 55, 91, 140, ...; in general the n^{th} one is $\frac{1}{6}n(n+1)(2n+1)$.*

Why does this formula give an integer value?
Answer: Either n or $n+1$ must be even, so the product of the three numbers must be divisible by 2. If either of these numbers is a multiple of 3, then of course it will be fine. If neither is, then n must be of the form $3m+1$, where m is an integer, and that means $n+1=3m+2$ and $2n+1=2(3m+1)+1=6m+3$, which is divisible by 3. So one of the numbers will always be divisible by 3, and one divisible by 2, so dividing by 6 must give an integer answer.

1.17.5 How many *small* triangles are there altogether in this drawing?



Answer: 25.
Start with fewer rows and look for a pattern.
Can count triangles with a side at the bottom (type 1) and triangles with a point at the bottom (type 2) separately (shade in one set).

no. of rows of triangles	***no. of triangles (type 1)***	***no. of triangles (type 2)***	***total no. of triangles***
1	*1*	*0*	*1*
2	*3*	*1*	*4*
3	*6*	*3*	*9*
4	*10*	*6*	*16*
5	*15*	*10*	*25*
6	*21*	*15*	*36*
7	*28*	*21*	*49*

so if there are n rows of small triangles, the total number of triangles = n^2.
So we see that the sum of two consecutive triangle numbers is a square number, or

$$\frac{n(n-1)}{2}+\frac{(n+1)n}{2}=\frac{n(n-1+n+1)}{2}$$

$$=\frac{2n^2}{2}=n^2$$

or $T_{n-1} + T_n = S_n$.

Another way to count them is row-by-row. This shows that the sum of the first n odd numbers is the n^{th} square number.

Because the large equilateral triangle is mathematically similar to the small ones, as the number of rows increases as n the area (the number of small equilateral triangles) increases as n^2.

How many triangles (of any size) are there altogether in the drawing?

(This is much harder.)

Answer: 48.
Again, start with a drawing with fewer rows and look for a pattern.

no. of rows of triangles	***no. of triangles (type 1)***	***no. of triangles (type 2)***	***total no. of triangles***
1	*1*	*0*	*1*
2	*4*	*1*	*5*
3	*10*	*3*	*13*
4	*20*	*7*	*27*
5	*35*	*13*	*48*
6	*56*	*22*	*78*
7	*84*	*34*	*118*

After n rows the number of type 1 triangles is the sum of the first n triangle numbers.

The sum of the 1st n triangle numbers is the n^{th} tetrahedral number $\frac{1}{6}n(n+1)(n+2)$ (see section 1.17.4).

The number of type 2 triangles is more complicated, because you only get a bigger type 2 triangle on every other *row (a bigger type 1 triangle appears with* every *new row).*
So the number of type 2 triangles depends on whether n is even (then it's the sum of the alternate triangle numbers starting with the first one (1)) or odd (then it's the sum of the alternate triangle numbers but beginning with the second one (3)).

For n even, the number of type 2 triangles is $\frac{1}{24}n(n+2)(2n-1)$.
For n odd, the number of type 2 triangles is $\frac{1}{24}(n+1)(2n^2+n-3)$.

Adding $\frac{1}{6}n(n+1)(n+2)$ to each of these gives the formulas for the total number of triangles when n is even and when n is odd.

So the total number of triangles is $\frac{1}{8}n(n+2)(2n+1)$ if n is even, and $\frac{1}{8}(n+1)(2n^2+3n-1)$ if n is odd.

1.17.6 Which integers from 1 to 20 can be made from the sum of just 2 triangle numbers?
(The triangle numbers themselves can obviously be

Answer:
All except 5, 8, 14, 17 and 19.

made from just one triangle number.)

no.	sum	no.	sum	no.	sum
1	= 1	**2**	= 1 + 1	**3**	= 3
4	= 3 + 1	**5**	= 3 + 1 + 1	**6**	= 6
7	= 6 + 1	**8**	= 6 + 1 + 1	**9**	= 6 + 3
10	= 10	**11**	= 10 + 1	**12**	= 6 + 6
13	= 10 + 3	**14**	= 10 + 3 + 1	**15**	= 15
16	= 15 + 1	**17**	= 15 + 1 + 1	**18**	= 15 + 3
19	= 15 + 3 + 1	**20**	= 10 + 10		

and other possibilities.

Gauss (1777-1855) proved that every *integer is the sum of at most three triangle numbers.*

Which numbers can be expressed as the sum of two *consecutive* triangle numbers?

4, 9, 16; i.e., the square numbers (see section 1.17.5).

1.17.7 Polygon Numbers.
Can you define pentagon, hexagon, heptagon, etc. numbers?
Find different formulas for the different n^{th} polygon numbers.

Answer:
Make the length of each side increase by 1 as n *goes up by 1.*
If p *is the number of sides of the polygon (so* $p = 3$ *for the triangle numbers), then the* n^{th} p*-gon number is* $\frac{1}{2}n(n(p-2)-(p-4))$ *, and for* $n = 1$ *this reduces to 1 no matter what the* p *value. So 1 is the first number in all the polygon number sequences.*

This can also be written as $\frac{1}{2}n(np-2n-p+4)$ *.*

What kind of numbers are “rectangle numbers”?
If you put $p = 2$ *into the formula you just get* n *.*

Rectangle numbers would be either all the integers (if you allow rectangles with a side of length 1), or all non-prime numbers (if you don’t). (Non-primes are called “composites”.)

1.17.8 Mystic Rose.
Space four points evenly around the circumference of a circle. Join every point to every other point.
How many lines do you need?

Answer: 6 (four sides of the square plus two diagonals)
For n *points, the number of lines is the* $(n-1)^{th}$ *triangle number,* $\frac{1}{2}n(n-1)$ *. This happens because each of the* n *points is connected to the other* $n-1$ *points, making* $n(n-1)$ *lines, but this double-counts each line (from both ends) so we put in the factor of* $\frac{1}{2}$ *.*

With more points it makes beautiful patterns that are suitable for display work.

Finding the number of regions is deceptively difficult – see section 1.19.11.

1.17.9 When we use the formula $\frac{1}{2}n(n+1)$ (where n is an integer), why do we always get an integer answer? (Picking just any two integers and multiplying them and dividing by 2 won’t necessarily give an integer.)

n *and* $n+1$ *are* consecutive *numbers, and that means that one is always odd and the other even. Therefore you’re always multiplying an odd number by an even number, and that always gives an even answer (multiplying any integer by an even number always gives an even answer). So when you halve the answer you get an integer.*
Another way to think about it is that $\frac{1}{2}n(n+1) = \frac{n}{2}(n+1) = n\frac{(n+1)}{2}$ *; i.e., you can halve either* n *or* $n+1$ *(whichever is even) and multiply the answer by the other one, so you’ll always get an integer.*

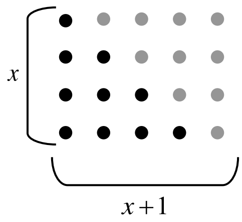
1.17.10 Prove that 8 times any triangle number is 1 less than a square number.

Answer: $8\times\frac{1}{2}n(n+1) = 4n(n+1) = 4n^2+4n$ *, which is 1 less than* $(2n+1)^2$ *, so it’s always 1 less than an odd square.*

1.17.11 Consecutive Sums.
What is the total of all the integers from 1 to 100? What about 1 to 1000? Or more?

You can tell the story of Gauss (1777-1855) who was told by a teacher to add up all the numbers from 1 to 100 (without a calculator, of course, in those days) and did it very quickly.

Pupils may think that it ought to be possible to replace repeated addition by some sort of multiplication, and that is what we're doing.



Answers: $S_{1-10}=55$, $S_{1-100}=5050$ *and* $S_{1-1000}=500500$ *, etc.*
(where S_{a-b} *means the sum of all the integers from* a *to* b *inclusive; i.e.,* $\sum_{i=a}^{b} i$ *).*

Various approaches:

1. *Realise by drawing dots or squares, that* S_{1-x} *is the* x^{th} *triangle number. Use the formula* $\frac{1}{2}x(x+1)$ *or see this by combining two identical triangles of dots and getting a rectangle* x *by* $(x+1)$.
2. *Say that the "average" value (actually the mean and median) of the numbers from 1 to* x *must be* $\frac{1+x}{2}$ *, and since there are* x *values the total must be* $\frac{x(x+1)}{2}$.
3. *Pair up 1 and* x *(to make* $x+1$*), 2 and* $x-1$ *(to make* $x+1$ *also), etc. Eventually you have* $\frac{1}{2}x$ *pairings (if* x *is even), so the total is* $\frac{x(x+1)}{2}$. *(But what if* x *is odd? Then there are* $\frac{x-1}{2}$ *pairings and the middle number (* $\frac{x+1}{2}$ *) is left over. So the total this time is* $\frac{x-1}{2}\times(x+1)+\frac{x+1}{2}=\frac{x+1}{2}(x-1+1)=\frac{x(x+1)}{2}$ *, the same.)*
4. *Find* $S_{1-10}=55$ *by some method (or just add them up) and argue that* S_{11-20} *must be 100 more, because each number in the sum is 10 more than each number in the first sum (writing out some of it makes this clearer). So altogether* S_{11-20} *must be* 10×10 *bigger* $=155$. *Now* $S_{1-20}=55+155=210$, *so* $S_{21-40}=210+20\times20=610$ *(by similar reasoning), and so* $S_{1-40}=820$, *and so on, giving* $S_{41-80}=2420$, $S_{1-80}=3240$, $S_{81-100}=1810$ *(* $=S_{1-20}+20\times80$ *), so* $S_{1-100}=3240+1810=5050$. *Although this method is not quick, it does involve some good thinking.*

Which integers is it possible to make using the sum of consecutive positive integers?

See sheet "Sums of Consecutive Integers".
Clearly all odd numbers (except 1) are possible (see diagonal line in the table), since pairs of consecutive integers added together make all of the odd numbers.
Impossible totals are 1, 2, 4, 8, 16, ... (powers of 2).
You can see this because in the formula $T_b-T_a=\frac{1}{2}(b-a)(a+b+1)$ *, one bracket must be even and one odd, so* T_b-T_a *has at least one odd factor, and so can't be a power of 2.*

Sums of Consecutive Integers

The sums of consecutive integers are the *trapezium numbers*, which are the differences between (non-consecutive) triangle numbers.

This happens because $\sum_{x=1}^{a} x$ is the a^{th} triangle number (T_a) and $\sum_{x=1}^{b} x$ is the b^{th} triangle number (T_b). [$T_n = \frac{1}{2}n(n+1)$]

So $\sum_{x=a+1}^{b} x$ (the sum of the consecutive integers from $a+1$ to b) is $T_b - T_a$, where $a \geq 0$ and $b - a > 1$. [$T_b - T_a = \frac{1}{2}(b-a)(a+b+1)$]

The possible integer sums are given in *italics* in the table below (horizontal triangle number minus vertical triangle number).

		T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}	T_{11}	T_{12}	T_{13}	T_{14}	T_{15}	T_{16}	T_{17}	T_{18}	T_{19}	T_{20}
		1	***3***	***6***	***10***	***15***	***21***	***28***	***36***	***45***	***55***	***66***	***78***	***91***	***105***	***120***	***136***	***153***	***171***	***190***	***210***
T_1	**1**			*5*	*9*	*14*	*20*	*27*	*35*	*44*	*54*	*65*	*77*	*90*	*104*	*119*	*135*	*152*	*170*	*189*	*209*
T_2	**3**				*7*	*12*	*18*	*25*	*33*	*42*	*52*	*63*	*75*	*88*	*102*	*117*	*133*	*150*	*168*	*187*	*207*
T_3	**6**					*9*	*15*	*22*	*30*	*39*	*49*	*60*	*72*	*85*	*99*	*114*	*130*	*147*	*165*	*184*	*204*
T_4	**10**						*11*	*18*	*26*	*35*	*45*	*56*	*68*	*81*	*95*	*110*	*126*	*143*	*161*	*180*	*200*
T_5	**15**							*13*	*21*	*30*	*40*	*51*	*63*	*76*	*90*	*105*	*121*	*138*	*156*	*175*	*195*
T_6	**21**								*15*	*24*	*34*	*45*	*57*	*70*	*84*	*99*	*115*	*132*	*150*	*169*	*189*
T_7	**28**									*17*	*27*	*38*	*50*	*63*	*77*	*92*	*108*	*125*	*143*	*162*	*182*
T_8	**36**										*19*	*30*	*42*	*55*	*69*	*84*	*100*	*117*	*135*	*154*	*174*
T_9	**45**											*21*	*33*	*46*	*60*	*75*	*91*	*108*	*126*	*145*	*165*
T_{10}	**55**												*23*	*36*	*50*	*65*	*81*	*98*	*116*	*135*	*155*
T_{11}	**66**													*25*	*39*	*54*	*70*	*87*	*105*	*124*	*144*
T_{12}	**78**														*27*	*42*	*58*	*75*	*93*	*112*	*132*
T_{13}	**91**															*29*	*45*	*62*	*80*	*99*	*119*
T_{14}	**105**																*31*	*48*	*66*	*85*	*105*
T_{15}	**120**																	*33*	*51*	*70*	*90*
T_{16}	**136**																		*35*	*54*	*74*
T_{17}	**153**																			*37*	*57*
T_{18}	**171**																				*39*